## NOTES FOR 16 JAN (THURSDAY)

### 1. Recap

(1) Proved Sobolev embedding and compactness
(2) Defined elliptic operators and constructed a parametrix.

### 2. Constant-coefficient elliptic operators on the torus

*Proof.* (1) Note that $|\widehat{\vec{u}_{app}}(\vec{k})| \leq C\frac{\|\widehat{\vec{f}}(\vec{k})\|}{\|\vec{k}\|^l}$ if $|\vec{k}| \geq N$ where $N$ is sufficiently (depending on the ellipticity constants and the coefficients of the lower order terms) large. Indeed, the magnitude of the lower order terms is less than $C(\|\vec{k}\|^{l-1} + \|\vec{k}\|^{l-2} + \ldots \leq C\|\vec{k}\|^{l-1})$ if $\|\vec{k}\| > 1$. Now $\|[\sigma_{\vec{k}} + lower][\vec{v}]\| \geq (\delta_1\|\vec{k}\|^l - C\|\vec{k}\|^{l-1})\|\vec{v}\|$. Of course if $|\vec{k}| \geq N$ is large, then $\|\hat{L}[\vec{v}]\| \geq c\|\vec{v}\|$ where $c > 0$. Hence $\|\hat{L}^{-1}[\vec{v}]\| \leq C\|\vec{k}\|^{-l}\|\vec{v}\|$ for large $N$.

The above easily implies that $\vec{u}_{app} \in H^{s+l}$. Moreover, $\|\vec{u}_{app}\|_{H^{s+l}} \leq C\|f\|_{H^s}$.

(2) The last inequality implies this result.

(3) $K(f) = L \circ G(f) - f = L(u_{app}) - f = -\sum_{|k|<N} \widehat{\vec{f}}(\vec{k})e^{i\vec{k}.\vec{x}}$. Now $K(f)$ is smooth and is hence in $H^a$ $\forall$ $a$. By the Rellich compactness lemma, $K(f) : H^s \to H^s$ is compact. Now $G(L(u)) - u = -\sum_{|k|<N} \hat{u}(k)e^{i\vec{k}.\vec{x}}$. As before this is a smooth function and hence by the Rellich lemma, $G \circ L - I$ is compact.

(4) Taking Fourier series on both sides, $\hat{L}\widehat{\vec{u}}(\vec{k}) = \widehat{\vec{f}}(\vec{k})$. Of course, for large $|k|$, $u$ coincides with $u_{app}$. For small $|k| < N$, $(1+|k|)^{s+l} \leq (1+N)^{s+l} \leq C$ where $C$ depends only on $N, s, l$ and hence only on the ellipticity constants, $s, l$, and the bounds on the lower order coefficients. This proves the result.

□

Now we define a useful notion in functional analysis.

**Definition 2.1.** Suppose $H_1, H_2$ are Hilbert spaces. A bounded linear operator $T : H_1 \to H_2$ is called Fredholm if $ker(T), Coker(T)$ are finite-dimensional.

We prove the following useful theorem about Fredholm operators. In these results, we use the easy fact that if $T$ is a bounded linear operator and $K$ is compact, then $T \circ K$ and $K \circ T$ are compact. We also use a slightly more difficult fact that if $K$ is compact, then $K^*$ is so as well.

**Theorem 2.2.** (1) *If $Im(T)$ is closed, then $Coker(T)$ is naturally a Banach space isomorphic to $Im(T)^{\perp}$. Therefore, $Coker(T)$ is a Hilbert space.*
(2) *If the range of $T$ is closed, then $Coker(T)^* \simeq Ker(T^*)$ where $T^* : H_2^* \to H_1^*$.*
(3) *If $ker(T), Coker(T)$ are finite dimensional, then the range is closed.*
(4) *$T$ is Fredholm if and only if $T^*$ is so.*
(5) *$T$ is Fredholm if and only if there exist bounded linear maps $G_1, G_2 : H_2 \to H_1$, such that $G_1 \circ T - I$, $T \circ G_2 - I$ are compact operators.*

(6) *The set of Fredholm operators $S \subset B(H_1, H_2)$ is open.*
(7) *Suppose $I \subset \mathbb{R}$ is a connected set. If $F(t) : I \subset \mathbb{R} \to S$ is a continuous map, then the index $Ind(F(t)) = dim(Ker(F(t))) - dim(Coker(F(t)))$ is a constant.*
(8) *If $K : H_1 \to H_2$ is a compact operator and $T$ is Fredholm, then $T + K$ is Fredholm with the same index.*

*Proof.* (1) Define $\|[y]\| = \inf_{y \in [y]} \|y\|$. By Riesz's lemma, the infimum is attained as a minimum $y_0 \in Im(T)^{\perp}$. The map $[y] \to y_0$ is linear and an isomorphism. We are done.

(2) Take $\rho \in ker(T^*) \subset H_2^*$ to $\lambda \in Coker(T)^*$ where $\lambda([y]) = \rho(y)$. This map $V : ker(T)^* \to Coker(T)^*$ is well-defined because $\rho(Tx) = T^*(\rho)(x) = 0$ by definition. It is clearly a linear map (and bounded). If the range is closed, then $Coker(T)$ is a Hilbert space isomorphic to $Im(T)^{\perp}$. Consider the map $U : Coker(T)^* \to H_2^*$ given by $U(\lambda)(v) = \lambda([v])$. This map is clearly linear and bounded. It can be easily seen to invert $V$.

(3) Mistake in the proof. (To be corrected next time.)

(4) If $T$ is Fredholm, then $T : ker(T) \oplus ker(T)^{\perp} \to Coker(T) \oplus Im(T)$ is bounded linear and defines an injective map $T_1 : ker(T)^{\perp} \to H_2$. Define $G(a \oplus b) = T_1^{-1}(b)$. Clearly, $G \circ T - I$ is a projection onto a finite dimensional subspace and hence compact. Now $T \circ G(a \oplus b) - a \oplus b = T(T_1^{-1}(b)) - a \oplus b = -a \oplus 0$ which is another projection and hence compact.

Conversely, if there exists such $G_1, G_2$, then $G_1 T = I + K$. Therefore $Ker(T) \subset Ker(G_1 T) = Ker(I + K)$ which we claim is finite-dimensional. Indeed, if $v_i$ is a bounded sequence in $Ker(I + K)$, then $Kv_i = -v_i$ has a convergent subsequence. But the unit ball is compact in a Banach space if and only if the space is finite-dimensional (Riesz's lemma). Thus $ker(T)$ is finite dimensional. Likewise, $Coker(T)$ is finite dimensional : Mistake in the proof. Will correct the next time.

□