NOTES FOR 23 JAN (THURSDAY)

1. Recap

(1) Proved a functional analytic theorem about Fredholm operators.
(2) Proved that elliptic operators are Fredholm and that they satisfy elliptic regularity.

2. Riemannian manifolds and metrics on vector bundles

In order to define $\Delta u = f$ on a manifold, unfortunately, we cannot do this locally by choosing coordinates and saying $\sum_i \frac{\partial^2}{\partial (x^i)^2} u = f$ because if we change coordinates, then the PDE will not be the same. So how can hope to even set up the Poisson PDE on a manifold ?

Another way of looking at the Laplacian is $\Delta = \nabla . \nabla$. So if we can define a dot product on every tangent space, and define the ∇ operation, then we can define the Laplacian. Why would we care about defining the Laplacian ? Among other things, it gives insight into the De Rham cohomology of the manifold.

Recall that a smooth vector bundle V over a smooth manifold M is a "smoothly varying collection of vector spaces parametrised by M", i.e., locally, $V \simeq U \times \mathbb{R}^r$ (where instead of $\mathbb{R}$, we can also have $\mathbb{C}$ - such a beast is a complex vector bundle) via a trivialisation, i.e., a collection of smooth sections $e_1, \ldots, e_r : U \subset M \to V$ such that $e_1(p), \ldots, e_r(p)$ form a basis for V_p at all $p \in U$. Equivalently, a vector bundle is simply a collection $(U_\alpha, g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r))$ satisfying $g_{\alpha\alpha} = Id, g_{\alpha\beta} = g_{\beta\alpha}^{-1}, g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = Id$. Fundamental examples of vector bundles are the tangent bundle TM, the cotangent bundle T^*M, and the bundles of differential forms $\Omega^k(M)$. These bundles can be defined using transition functions. A smooth section $s : M \to V$ is a smooth function satisfying $\pi \circ s = Id$.

A metric g on a vector bundle V over M is a smooth section of $V^* \otimes V^*$ such that on each fibre it is symmetric and positive-definite. In other words, suppose e_i is a trivialisation of V over U and e^{i*} the dual trivialisation of V^* over U, then[1] $g(p) = g_{ij}(p) e^{i*} \otimes e^{j*}$ where $g_{ij} : U \subset M \to GL(r, \mathbb{R})$ is a smooth matrix-valued function to symmetric positive-definite matrices. So a metric is simply a smoothly varying collection of dot products, one for each fibre. Using a partition-of-unity one can prove the following result.

Theorem 2.1. *Every rank-r real vector bundle V over a manifold M admits a smooth metric g.*

In the special case when $V = TM$, the metric is called a Riemannian metric on M. If (x, U) is a coordinate chart, then $g(x) = g_{ij}(x) dx^i \otimes dx^j$. By symmetry, $g_{ij} = g_{ji}$. Moreover, g is a positive definite matrix. If one changes coordinates to y^μ then $g_{\mu\nu} = g_{ij} \frac{\partial x^i}{\partial y^\mu} \frac{\partial x^j}{\partial y^\nu}$. Given a metric g on TM, we get one on T^*M given by $g^* = g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$. Note that $g_{ik} g^{kj} = \delta_i^j$.

If M is oriented, supposing (x, U) is an oriented coordinate chart, then $vol = \sqrt{\det(g_{ij})} dx^1 \wedge dx^2 \ldots dx^m$ is a well-defined top form. Indeed, if we changes coordinates, it transforms correctly as seen in the linear algebra above. This form is called the "volume" form of the metric.

Here are examples :

[1] We will be using the Einstein summation convention. Repeated indices are understood to be summed over.

(1) Euclidean space $\mathbb{R}^n, g_{Euc} = \sum dx^i \otimes dx^i$. This is the usual metric. The length of a tangent vector v is $\sum (v^i)^2$.
(2) If we take the same Euclidean space $\mathbb{R}^2$ and use polar coordinates, $x = r\cos(\theta)$, $y = r\sin(\theta)$, then $dx = dr\cos(\theta) - r\sin(\theta)d\theta$, $dy = dr\sin(\theta) + r\cos(\theta)d\theta$. Thus, $g_{Euc} = dr \otimes dr + r^2 d\theta \otimes d\theta$.
(3) The circle $S^1 : g = d\theta \otimes d\theta$.
(4) If M, g_M, N, g_N are two Riemannian manifolds, then $M \times N, g_M \times g_N$ given by $g_M \times g_N(v_M \oplus v_N, w_M \oplus w_N) = g_M(v_M, w_M) + g_N(v_N, w_N)$. This gives a metric on the n-torus using the circle metric.
(5) The Hyperbolic metric $\mathbb{H}^m, g_{Hyp} : g_{Hyp} = \frac{\sum dx^i \otimes dx^i}{(x^m)^2}$.