NOTES FOR 25 FEB (TUESDAY)

1. Recap

(1) Stated the Hodge theorem and did two applications - Poincaré duality and the Kunneth formula.
(2) Defined Sobolev spaces of sections of a vector bundle on compact manifolds and proved equivalence under change of metrics and connections.

2. Sobolev spaces on general manifolds

To make another definition, we need a lemma :

Lemma 2.1. *If $\vec{s} : U \subset \mathbb{R}^m \to \mathbb{R}^r$ is in L^1_{loc} and weakly differentiable with weak derivatives $\partial_i \vec{s} = \vec{t}_i$, then for any smooth functions $g : U \to GL(r, \mathbb{R})$, diffeomorphisms $y(x) : U \to U$, the function $\vec{\tilde{s}} = g\vec{s}$ is weakly differentiable with weak derivative $\frac{\partial \vec{\tilde{s}}}{\partial y^i} = \frac{\partial g(x(y))}{\partial y^i} g^{-1} \vec{\tilde{s}} + g\vec{t}_j \frac{\partial x^j}{\partial y^j}$. (Note that this coincides with what we expect if $\vec{s}$ is smooth.)*

Proof. Indeed, if $\vec{\phi}$ is a smooth function with compact support in U, then

$$\int_U \Big(\langle \frac{\partial g(x(y))}{\partial y^i} g^{-1}\vec{\tilde{s}} + g\vec{t}_j \frac{\partial x^j}{\partial y^j}, \phi\rangle dy = \int_U \langle \frac{\partial g(x(y))}{\partial y^i} g^{-1} \vec{\tilde{s}}, \phi\rangle + \frac{\partial x^j}{\partial y^j}\langle \vec{t}_j, g^T\phi\rangle\Big) dy$$

$$= \int_U \langle \vec{\tilde{s}}, (\frac{\partial g(x(y))}{\partial y^i} g^{-1})^T \phi\rangle dy - \int_U \langle \vec{\tilde{s}}, (g^{-1})^T \frac{\partial}{\partial x^j}\left(\sqrt{\det\left(\frac{\partial \vec{y}}{\partial \vec{x}}\right)} \frac{\partial x^j}{\partial y^j} g^T \phi\right)\rangle \sqrt{\det\left(\frac{\partial \vec{x}}{\partial \vec{y}}\right)} dy$$

$$= -\int_U \langle \vec{\tilde{s}}, \frac{\partial \phi}{\partial y^i}\rangle dy - \int_U \langle \vec{\tilde{s}}, \frac{\partial}{\partial x^j}\left(\sqrt{\det\left(\frac{\partial \vec{y}}{\partial \vec{x}}\right)} \frac{\partial x^j}{\partial y^j}\right)\phi\rangle \sqrt{\det\left(\frac{\partial \vec{x}}{\partial \vec{y}}\right)} dy = -\int_U \langle \vec{\tilde{s}}, \frac{\partial \phi}{\partial y^i}\rangle dy \tag{2.1}$$

□

This shows that the notion of weak differentiability of an L^1_{loc} section of a vector bundle is well-defined in terms of coordinates and trivialisations.

Lemma 2.2. *Suppose (E, ∇, h) is a bundle with a metric and a compatible connection on (M, g) where M is any orientable manifold (not necessarily compact). Let $s \in L^1_{loc}(M)$ be a weakly differentiable section. Then the weak derivative ∇s is well-defined as an L^1_{loc} section of $T^*M \otimes E$ and satisfies $(\nabla s, \phi)_{L^2} = (s, \nabla^\dagger \phi)_{L^2}$ where ϕ is any compactly supported smooth section on M and $\nabla^\dagger$ is given by the same formula as before. Conversely, if this property is satisfied, then s is weakly differentiable (in the sense defined before).*

Proof. Define ∇s locally as $\frac{\partial \vec{s_\alpha}}{\partial x^i} dx^i + A_\alpha \vec{s}_\alpha$ where the derivatives are weak derivatives. From the previous lemma it is easily seen that it transforms like a section of $T^*M \otimes E$.

Suppose we cover M by a locally-finite cover U_α of charts which are also trivialising neighbourhoods, and we let ρ_β be a partition-of-unity subordinate to it (Note that ρ_β has compact support in some U_β but the indexing set need not be the same.) Then $(\nabla s, \phi) = \sum_\beta (\nabla s, \rho_\beta \phi)$ (the sum is finite because ϕ has compact support). Now $(\nabla s, \phi) = -\sum_\beta (s, d^\dagger(\rho_\beta \phi)) + \sum_\beta (s, A^\dagger \rho_\beta \phi) =$

$-\sum_\beta (s, \nabla^\dagger(\rho_\beta \phi)) = -\sum_\beta (s, \nabla^\dagger \phi)$ (where we used the property that $\nabla^\dagger$ is a first order differential operator and $d(\sum \rho_\beta) = 0$).

The converse part follows by taking ϕ to be supported in a coordinate trivialising open set. □

Now we define the Sobolev space in another way.

Definition 2.3. Suppose (E, ∇, h) is a bundle with a metric and a compatible connection on a compact oriented (M, g). Let $s \geq 0$ be an integer. Then the space $\tilde{H}^s_{\nabla,h,g}$ consists of s times weakly differentiable sections $\in L^2$ with inner product $(a, b) = \displaystyle\int \langle a, b\rangle vol_g + \langle \nabla a, \nabla b\rangle vol_g + \ldots$ where the derivatives are weak derivatives.

Lemma 2.4. *$\tilde{H}^s_{\nabla,h,g}$ is a Hilbert space and smooth sections are dense in it. Hence it coincides with $H^s_{\nabla,h,g}$.*

Proof. Hilbert space : If f_n is a Cauchy sequence, then ρf_n is also a Cauchy sequence for any smooth function ρ. Assume that ρ is compactly supported in a coordinate trivialising neighbourhood U. Thus ρf_n can be extended smoothly to $S^1 \times S^1 \ldots$ (by simply taking a large cube in $\mathbb{R}^m$ containing its support and periodically extending it). Moreover, it is also clear that ρf_n is Cauchy in $H^s(S^1 \times S^1 \ldots)$. Hence, $\rho f_n \to u$ for some $u \in H^s(S^1 \times S^1 \ldots)$. This function u has support in the previously chosen large rectangle and hence can be extended to all of M. Moreover, since the Sobolev norms are equivalent, this convergence happens in $H^s_{\nabla,h,g}$. $f_n = \sum \rho_\alpha f_n \to \sum u_\alpha$ in H^s where ρ_α is a partition-of-unity.
Smooth functions are dense : Suppose $\rho_\alpha \geq 0$ is such that $\sum \rho_\alpha^2 = 1$ and these are subordinate to a finite trivialising coordinate cover U_α. Suppose $f \in H^s_{\nabla,h,g}$. Then there are sequences of smooth functions $f_{n,\alpha} \to \rho_\alpha f$ in $H^s(S^1 \times S^1 \ldots)$. Now $\rho_\alpha f_{n,\alpha}$ is well-defined on M. Moreover, $\| \sum \rho_\alpha f_{n,\alpha} - \rho_\alpha \rho_\alpha f \|_{H^s_{\nabla,h,g}} \leq C \sum_\alpha \| \sum f_{n,\alpha} - \rho_\alpha \rho_\alpha f\|_{H^s(S^1 \times S^1 \ldots)} \to 0$. □

There is yet another way to define the Sobolev space.

Definition 2.5. Choose a finite cover of trivialising coordinate neighbourhoods $(U_\alpha, x^i_\alpha, e_{j,\alpha})$ and a partition-of-unity subordinate to it. The space $H^{‘s}$ is the space of all L^1_{loc} sections a such that $\|a\|^2 = \|\rho_\alpha \vec{a}_\alpha\|_{H^s(S^1 \times S^1 \ldots)} < \infty$. The inner product between a and b is $\displaystyle\sum_\alpha (\rho_{al} \vec{a}_\alpha, \rho_\alpha \vec{b}_\alpha)_{H^s}$

Lemma 2.6. *(Exercise) The space $H^{‘s}$ is well-defined independent of choices. It is a Hilbert space and smooth sections are dense in it. On smooth functions the $H^{‘s}$ norm is equivalent to the $H^s_{\nabla,h,g}$ norm with respect to any connection and hence it is homeomorphically isomorphic to $H^s_{\nabla,h,g}$.*

3. Sobolev embedding and compactness

Define $C^{k,\alpha}(M, E)$ as the space of C^k sections of E such that in local coordinates (and frames) they are $C^{k,\alpha}$. The norm on this space is $\|u\|_{C^{k,\alpha}} = \displaystyle\sum_\mu \|\vec{u}_\mu\|_{C^{k,\alpha}(\bar{U}_\mu)}$. This is independent of choices made and is a Banach space. This will be given as a HW problem.

Actually, this is equivalent to the norm $\sum \|\rho_\mu \vec{u}_\mu\|_{C^{k,\alpha}(\bar{U}_\mu)}$:

Proof. Indeed, firstly, $\sup_x |f(x)g(x)| + \sup_{x,y} \frac{|f(x)g(x) - f(y)g(y)|}{|x-y|^\alpha} \leq \|f\|_{C^{0,\alpha}} \|g\|_{C^{0,\alpha}}$. Hence $\sum \|\rho_\mu \vec{u}_\mu\|_{C^{k,\alpha}(\bar{U}_\mu)} \leq C\|u\|_{C^{k,\alpha}}$.

Next, if one changes coordinates and trivialisations, the resulting $C^{k,\alpha}$ norms are equivalent (a part of the the HW problem). Therefore, $\|\vec{u}_\mu\|_{C^{k,\alpha}(\bar{U}_\mu)} \leq \sum_{\nu \neq \mu} \|\rho_\nu \vec{u}_\mu\| + \|\rho_\mu \vec{u}_\mu\|$. Now $\|\rho_\nu \vec{u}_\mu\|_{C^{k,\alpha}(\bar{U}_\mu)} =$

$\|g_{\nu\mu}\rho_\nu\vec{u}_\nu\|_{C^{k,\alpha}(\bar{U})_\mu} \leq C\|\rho_\nu\vec{u}_\nu\|_{C^{k,\alpha}(\bar{U}_\nu)}$ where the last norm is in the ν coordinates. Hence we are done. □

Firstly, we have the following compactness result :

Lemma 3.1. *Suppose $k \leq l$. If $k < l$ or $0 \leq \beta < \alpha < 1$, then $C^{l,\alpha} \subset C^{k,\beta}$ is a compact embedding.*

Proof. The embedding part is trivial. We shall prove that $C^{0,\alpha} \subset C^0$ is compact (the general case is similar). Let ρ_α be a partition of unity. If $\|f_n\|_{C^{0,\alpha}} \leq C$, then $\|\rho_\mu f_n\|_{C^{0,\alpha}(\bar{U}_\mu)} \leq C$. By the usual Arzela-Ascoli argument, there is a subsequence (which we shall denote by f_n still) such that $\rho_\mu f_n \to f_\mu$ on $C^{0,\alpha}(\bar{U}_\mu)$ for some function $f_\mu : U_\alpha \to \mathbb{R}^r$. (For each μ there is a potentially different subsequence. We choose one for the first μ, then choose a further subsequence for the second μ and so on. There are only finitely many μ.) Clearly f_μ has compact support in U_μ and hence can be extended to a $C^{0,\alpha}$ section of E on M. Now $\|\sum_\mu f_\mu - f_n\|_{C^{0,\alpha}(M)} \leq C\sum_\mu \|f_\mu - \rho_\mu f_n\|_{C^{0,\alpha}(\bar{U}_\mu)} \to 0$. □

Now we prove Sobolev embedding plus compactness.

Theorem 3.2. *The following inclusions are compact. (Sometimes, this along with the above theorem are referred to as the Sobolev embedding theorems.)*

(1) *$H^s(E) \subset H^l(E)$ if $l < s$. (Rellich lemma.)*
(2) *$H^s(E) \subset C^a(M,E)$ if $s \geq [\frac{n}{2}] + a + 1$. (Rellich-Kondrachov compactness.)*

Proof.

(1) The inclusion part is clear. If f_n is a bounded sequence in $H^s(E)$, then $\rho_\alpha f_n \in H^s(S^1 \times S^1 \ldots)$ is a bounded sequence and by the usual Rellich lemma, it has a convergent subsequence (which abusing notation as usual we still denote by the subscript n) $\rho_\alpha f_n \to f_\alpha$ in $H^s(S^1 \times S^1 \ldots)$. Passing to a further subsequence (that converges a.e.) we see that f_α has support in U_α and hence can be thought of as being a global section on M. By equivalence of norms, $\rho_\alpha f_n \to f_\alpha$ in $H^s(M,E)$. Thus $\sum \rho_\alpha f_n = f_n \to \sum f_\alpha$.
(2) If $f \in H^s(E)$ then $\rho_\alpha f \in H^s(S^1 \times S^1 \ldots)$. Thus $\rho_\alpha f \in C^a(S^1 \times S^1 \ldots)$ by the usual Sobolev embedding on the torus. Hence, $\rho_\alpha f \in C^a(M,E)$ by equivalence of norms. Thus $\sum_\alpha \rho_\alpha f = f \in C^a(M,E)$. Likewise, if $f_n \in H^s(E)$ is bounded, then a subsequence $\rho_\alpha f_n \to f_\alpha$ in $C^a(S^1 \times S^1 \ldots)$. Since f_α is supported on U_α, as before $f_n = \sum \rho_\alpha f_n \to \sum f_\alpha$ in $C^a(M,E)$. □

4. Elliptic operators - Regularity

Now we define the notion of a uniformly elliptic operator : Suppose $(E, h_E, \nabla_E), (F, h_F)$ are smooth bundles with metrics and a metric compatible connection for E on a compact oriented (M,g) where TM is equipped with the Levi-Civita connection. Whenever we use ∇ in what follows, it is made out of ∇_E, ∇_g (Fix h_E, h_F, ∇_E, and g in whatever follows.) First we prove a "structure theorem" for linear PDOs.

Lemma 4.1. *To every linear PDO L of order o with smooth coefficients, there exist smooth maps $a_k : T^*M \otimes T^*M \otimes \ldots T^*M \otimes E \to F$ (where T^*M is repeated k times) such that $L(u) = \displaystyle\sum_{k=0}^{o} a_k \nabla^k u$.*

Proof. We prove this by induction on o. For $o = 0$, by tensoriality, there is such an endomorphism. Assume the result for $0, 1, \ldots, o-1$. Then locally, in a trivialising coordinate chart, $L(u)_\alpha = \sum_{k=0}^{o} a^I_{k,\alpha}\partial_I \vec{u}_\alpha$. If we change the trivialising coordinate chart, then $\vec{u}_\beta = g_{\beta\alpha}\vec{u}_\alpha$, and $\frac{\partial}{\partial y^i} = \frac{\partial x^j}{\partial y^i}\frac{\partial}{\partial x^j}$ (and the tensor product version of this). The highest order term changes as $a^I_{0,\alpha}\partial_{x,I}\vec{u}_\alpha \to$

$a^I_{o,\alpha} g_{\beta\alpha} \frac{\partial y^J}{\partial x^I} \partial_{y,J} \vec{u}_\beta$, i.e., a_o is a global section of $End(T^*M \otimes T^*M \ldots E, F)$. Hence $L(u) - a_o \nabla^o u$ is a linear PDO of order $o-1$ and hence by induction we are done. □

The formal adjoint L^*_{form} of L is defined as being a linear PDO of the same order given by $\sum_{k=0}^{o} (\nabla^k)^\dagger \circ a_k^\dagger$. It satisfies (and is equivalent to) $(L^*_{form} u, v) = (u, Lv)$ for smooth u, v.

Definition 4.2. The principal symbol of L is the Endomorphism $\sigma(L) : T^*M \otimes \ldots E \to F$ given by $\sigma(L) = a_o$. A linear PDO L with smooth coefficients is called uniformly elliptic with ellipticity constants $\delta_1, \delta_2 > 0$ if $\delta_1 |v|^2_{h_E(p)} \leq |\sigma_p(L)(\zeta, \zeta, \ldots, \zeta) v|^2_{h_F(p)} \leq \delta_2 |v|^2_{h_E(p)}$ $\forall\, p \in M, \zeta \neq 0 \in T^*_p M$ and the principal symbol is invertibel. (Please note that δ_1, δ_2 depend on the fixed h_F, h_E obviously.) In particular, the ranks of E and F are required to be the same.

It is clear that L is uniformly elliptic (from now on, called "elliptic") if and only if L^*_{form} is so. The ellipticity constants may be chosen to be equal. (Again, the ranks of E and F being the same is important for this.)

Definition 4.3. Suppose f is an L^2 section of F. An L^2 section u is said to be a distributional solution of $Lu = f$ if for every smooth section ϕ of F, $(u, L^*_{form}\phi) = (f, \phi)$. (Please note that we have not defined distributions in general. However, the notion of a distributional solution does not need distributions.)

Next we prove that distributional solutions of elliptic equations are smooth.

Theorem 4.4. *If L is uniformly elliptic and f a smooth section of F. Then if $u \in L^2$ satisfies $Lu = f$ in the sense of distributions then u is smooth. Moreover, if $f \in H^s$, then $u \in H^{s+l}$ and $\|u\|_{H^{s+o}} \leq C_s(\|f\|_{H^s} + \|u\|_{L^2})$ where C_s depends only on h_E, h_F, g, ∇_E, an upper bound on $\|a_k\|_{C^{s+o}}$, and on the ellipticity constants.*

We claim that this theorem follows from

Theorem 4.5. *If L is uniformly elliptic, u is a smooth section of E, then $\|u\|_{H^{s+o}} \leq C_s(\|Lu\|_{H^s} + \|u\|_{L^2})$.*

Indeed, assume this theorem. Then we shall prove theorem 4.4. Suppose u_n are smooth sections converging to u in L^2. Then $\|u_n\|_{H^{s+o}} \leq C_s(\|Lu_n\|_{H^s} + \|u_n\|_{L^2})$ according to theorem 4.4. Note that $(Lu_n, \phi) = (u_n, L^*_{form}\phi) \to (u, L^*_{form}\phi) = (f, \phi)_{L^2}$ $\forall\, \phi$. The family of functionals $T_n : \phi \in C^\infty(M, E) \to (Lu_n, \phi)$ is bounded for every ϕ because $|(Lu_n, \phi)| = |(u_n L^*_{form}\phi)| \to |(f, \phi)| \leq \|f\|_{L^2} \|\phi\|_{L^2}$. Moreover, this shows that T_n can be extended (in a norm preserving manner) to L^2 because smooth sections are dense. So $|T_n(\phi)| \leq \|f\| \|\phi\| + \epsilon$ for $n > N_{\phi,\epsilon}$. The family of functionals $\frac{T_n}{\|f\| + \epsilon}$ is pointwise bounded and hence uniformly bounded. Thus $\|T_n(\phi)\| \leq C(\|f\| + \epsilon)\|\phi\|$. Take $\phi = Lu_n$ to conclude that $\|Lu_n\|_{L^2} \leq C(\|f\| + \epsilon)$. Thus $\|u_n\|_{H^o} \leq C_s(\|f\|_{L^2} + \|u\|_{L^2}) + C\epsilon$. A bounded sequence in a Hilbert space has a weakly convergent subsequence (Banach-Alaoglu). Hence (upto a subsequence) u_n weakly converges in H^o to some function and that better be u (because u_n strongly converges to u in L^2). Thus $u \in H^o$. Moreover, $|(u, \phi)| \leq |(u_n, \phi)_o| \leq (C_s(\|f\|_{L^2} + \|u\|_{L^2}) + C\epsilon)\|\phi\|_o$. Since this is true for all ϵ, ϕ, we have the desired estimate on u for $s = 0$. Moreover, $Lu = f$ in the strong sense (and hence almost everywhere).

For higher values of s, we use induction. We only prove for $s = 1$ (the general inductive case is similar). It is easy to see that in the distributional sense, $L(\nabla u) = \nabla f - [\nabla, L]u$ where the right hand side is bounded in L^2 by $C(\|f\|_{H^1} + \|u\|_{L^2})$ (where we are using the $s = 0$ case). Now use $\nabla u \in H^o$ and satisfies the same estimate. Hence we get the $s = 1$ case and so on.