NOTES FOR 28 JAN (TUESDAY)

1. Recap

(1) Recalled definitions and examples of Vector bundles. Did some constructions of new vector bundles from old ones.
(2) Defined metrics, and gave examples.

2. Riemannian manifolds and metrics on vector bundles

Recall the definition of an induced metric

Definition 2.1. If g is a metric on M and $S \subset M$ is an embedded submanifold, then g induces a metric $g|_S$ on S given by $g_p|_S(v_S, w_S) = g_p(i_* v_S, i_* w_S)$.

(1) $S^2 \subset \mathbb{R}^3$. First write the metric in $\mathbb{R}^3$ in spherical coordinates $z = r\cos(\theta)$, $x = r\sin(\theta)\cos(\phi)$, $y = r\sin(\theta)\sin(\phi)$. Thus, $g_{Euc} = dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2(\theta) d\phi \otimes d\phi$. Now when we restrict to the unit sphere, the tangent vectors do not include $\frac{\partial}{\partial r}$. Thus, $g_{Sphere} = d\theta \otimes d\theta + \sin^2(\theta) d\phi \otimes d\phi$
(2) Suppose $z = f(x,y)$ is the graph of a function, then $g_{Induced} = dx \otimes dx + dy \otimes dy + (\frac{\partial f}{\partial x})^2 dx \otimes dx + (\frac{\partial f}{\partial y})^2 dy \otimes dy + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}(dx \otimes dy + dy \otimes dx)$.

Now we write down the volume forms of most of the above examples :

(1) $vol_{Euc} = dx^1 \wedge dx^2 \wedge \ldots dx^n$.
(2) In polar coordinates in $\mathbb{R}^2$, $vol_{Euc} = \sqrt{\det(g)} dr \wedge d\theta = r dr \wedge d\theta$.
(3) For the circle, $vol = d\theta$.

Suppose $\gamma : [0,1] \to M$ is a smooth path. Then, define its length as $L(\gamma) = \int_0^1 \sqrt{g(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})} dt$. A piecewise C^1, regular (meaning that $\gamma' \neq 0$ throughout) curve satisfying the following equation is called a geodesic.

$$\frac{d^2\gamma^r}{dt^2} + \Gamma^r_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0$$

$$\Gamma^r_{ij} = g^{rl}\frac{1}{2}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right) \tag{2.1}$$

It can be proved that every geodesic is actually smooth, can be parametrised by its arc-length, and that arc-length parametrised geodesics are precisely the critical points of the length functional. Moreover, the function $d(p,q) = \inf L(p,q)$ over all the piecewise C^1 paths joining p and q is a metric and that the topology induced by it is the same as the original topology of the manifold.

Now we note that geodesics exist locally, and that if γ is a geodesic, then so is $\gamma(ct)$. In fact, we have the following result.

Theorem 2.2. *Let $p \in M$. Then there is a neighbourhood U_o of p and a number $\epsilon_p > 0$ such that for every $q \in U$ and every tangent vector $v \in T_qM$ with $\|v\| < \epsilon_p$ there is a unique geodesic $\gamma_v : (-2,2) \to M$ satisfying $\gamma_v(0) = q$, $\frac{d\gamma_v}{dt}(0) = v$.*

If $v \in T_qM$ is a vector for which there is a geodesic, $\gamma : [0,1] \to M$ satisfying $\gamma(0) = q$ and $\gamma^{'}(0) = v$ then we define $\exp_q(v) = \gamma_v(1)$. The geodesic itself can be described as $\gamma(t) = \exp_q(tv)$ (by the uniqueness theorem for ODE). By the smooth dependence on parameters of an ODE, $\exp_q(v)$ depends smoothly on q and on v and defines a smooth map $\exp_q : T_qM \to M$.
Note that $(\exp_q)_{v*} : T_v(T_qM) \simeq T_qM \to T_{\exp_q(v)}M$ is its pushforward. We claim that

Theorem 2.3. *$(\exp_q)_{0*} = Id$ and hence $\exp_q$ is a local diffeomorphism around $\vec{0}$.*

Proof. Clearly the first statement and the inverse function theorem imply the second. Now if $v \in T_qM$, we need to obtain a curve $c(t) \in T_qM$ such that $c(0) = 0$, $c'(0) = v$, and $\frac{d\exp_q(c(t))}{dt}|_{t=0} = v$. Let $c(t) = tv$. Then $\exp_q(c(t)) = \exp_q(tv)$ which is the time-t geodesic starting at q pointing along v at $t = 0$. Thus we are done. □

In fact, we can say more.

Theorem 2.4. *Geodesics are locally length minimising. Moreover, if $p \in M$, there exists a geodesic ball $B_{\epsilon_p}(p)$ such that every two points in the ball can be connected by a unique length minimising geodesic lying in the ball and such that the exponential map is a diffeomorphism restricted to the ball. Such a ball is called a geodesically convex ball.*

Now we make a definition of a useful coordinate system.

Definition 2.5. Given $q \in M$, the coordinate system defined by $\exp_q : U \subset T_qM \to M$ is called a geodesic normal coordinate system at q (after choosing coordinates on U that is).

This set of coordinates is extremely useful. In fact,

Theorem 2.6. *There is a geodesic normal coordinate system v at p, $g_{ij}(p) = \delta_{ij}$ and $\frac{\partial g_{ij}}{\partial v^k}(p) = 0$.*

Proof. Choose coordinates x^μ so that $g_{\mu\nu}(p) = \delta_{\mu\nu}$. (This can be easily accomplished by taking any coordinate system and rotating it so as to diagonalise g.) Let v^i be coordinates in T_pM. Now exp is a local diffeomorphism. So $x^\mu(v^j) = x^\mu \circ \exp(v^j)$ is a change of coordinates in a small neighbourhood.

Note that since $\exp_{0*} = Id$, $\frac{\partial x^\mu}{\partial v^j}|_{v=0} = \delta^\mu_j$. Now $\tilde{g}_{ij} = g_{\mu\nu}\frac{\partial x^\mu}{\partial v^i}\frac{\partial x^\nu}{\partial v^j}$. So it is easy to see that $\tilde{g}_{ij}(0) = \delta_{ij}$. Since the geodesics through p are linear in this coordinate system, we see that the Christoffel symbols $\tilde{\Gamma}^r_{ij}(0) = 0$. It is easy to see that if the Christoffel symbols are 0, then so are all first partial derivatives of the metric. □

More generally, any coordinate system in which the metric at p is standard upto first order is called a normal coordinate system at p.

Actually, we can prove the existence of normal coordinates in much simpler manner even without reference to geodesics.

Theorem 2.7. *There is a normal coordinate system y at p.*

Proof. Choose any coordinate system at x at p such that $x = 0$ is p. Using a linear map, we may diagonalise g at p. So without loss of generality, $\tilde{g}_{\mu\nu} = \delta_{\mu\nu} + a_{\mu\nu\alpha}x^\alpha + O(x^2)$. (Note that $a_{\mu\nu\alpha} = a_{\nu\mu\alpha}$.) Change the coordinates to y such that $x(y)^i = y^i + b^i_{jk}y^jy^k$ where $b^i_{jk} = b^i_{kj}$. Now

$$
\begin{aligned}
g_{ij} = \tilde{g}_{\mu\nu}\frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j} &= (\delta_{\mu\nu} + a_{\mu\nu\alpha}y^\alpha + O(y^2))(\delta^\mu_i + b^\mu_{ik}y^k)(\delta^\nu_j + b^\nu_{jk}y^k) \\
&= \delta_{ij} + a_{ijk}y^k + (b_{ijk} + b_{jik})y^k + O(y^2) \qquad (2.2)
\end{aligned}
$$

So we just need to choose b so that $a_{ijk} = -b_{ijk} - b_{jik}\ \forall\ k$. So take $b = -\frac{a}{2}$. □

It is natural to ask if there is a geodesic normal coordinate system to the second order. Shockingly enough, there isn't (in general). In fact,

Theorem 2.8. *There exists a* $(0,4)$ *tensor (called the Riemann curvature tensor of* g*) which is locally* $R_{\mu\nu\alpha\beta}$ *such that in geodesic normal coordinates,*

$$g_{\mu\nu} = \delta_{\mu\nu} - \frac{1}{3}R_{\mu\alpha\nu\beta}(0)x^{\mu}x^{\nu} + O(x^3) \tag{2.3}$$

where in these coordinates, $R_{ijkl}(0) = \frac{1}{2}\frac{\partial^2 g_{jk}}{\partial x^i \partial x^l}(0) + \frac{1}{2}\frac{\partial^2 g_{il}}{\partial x^j \partial x^k}(0) - \frac{1}{2}\frac{\partial^2 g_{jl}}{\partial x^i \partial x^k}(0) - \frac{1}{2}\frac{\partial^2 g_{ik}}{\partial x^j \partial x^l}(0)$*. In fact, all the other terms in the Taylor expansion depend only on* R *and its derivatives. So there is a change of coordinates such that* g *is Euclidean everywhere, then, since the Euclidean coordinates are geodesically normal, the Riemann curvature tensor is identically* 0*.*

So one can prove that one cannot draw a map of any part of Bangalore on a piece of paper such that distances are to scale, by calculating the curvature of the sphere with the metric induced from the Euclidean space. It turns out to be a non-zero tensor. We will return to curvature later on in a different way. This theorem is to show you that the notion of curvature is “forced” upon us. (It is not an artificial definition.)