NOTES FOR 30 JAN (THURSDAY)

1. Recap

(1) Induced metrics on submanifolds and examples.
(2) Geodesics, properties, and normal coordinates.

2. Connections and curvature

Here are a bunch of observations / questions :

(1) In $\mathbb{R}^n$, you have the idea of a "constant" vector field. (Indeed, this is one way to prove that $\mathbb{R}^n$ is parallelizable, i.e., it has trivial tangent bundle.) So you need to able to find the directional derivative $\nabla_V X$ of any vector field along a direction V. Note that if we manage to define this concept, then $\nabla_{\gamma'(t)} X(\gamma(t)) = 0$ amounts to parallel transporting the vector field along γ.

(2) Suppose $(S, g_S) \subset (\mathbb{R}^n, Euc)$ is a submanifold with the induced metric. (Actually every Riemannian manifold is of this form by the Nash embedding theorem.) Suppose X is a tangent vector field along S. Suppose that $N_1, N_2 \ldots, N_k$ are local linearly independent unit normal vector fields on $U \subset S$ (where $k = n - dim(S)$). Assume that V is a tangent vector on S at p. How can we define the directional derivative $\nabla_V X(p)$? Clearly, the usual Euclidean directional derivative $D_V X = \frac{\partial \vec{X}}{\partial x^i} V^i$ is not the right one because it measures how fast X is changing perpendicular to S as well. So we need to project this back to S.

In other words, the "correct" way to define a directional derivative is $\nabla_V X = D_V X - \sum_{i=1}^{k} \langle D_V X, N_i \rangle_{Euc} N_i$. Now note that $\langle D_V X, N_i \rangle_{Euc} = D_V \langle X, N_i \rangle_{Euc} - \langle X, D_V N_i \rangle_{Euc} = -\langle X, D_V N_i \rangle_{Euc}$. In other words,

$$\nabla_V X = D_V X - \sum \langle X, D_V N_i \rangle_{Euc} N_i = D_V X + a\ term\ linear\ in\ X. \tag{2.1}$$

It turns out (miraculously) that the linear term is related to the Christoffel symbols and the Riemann curvature tensor of g_S that we defined before. This way of defining a directional derivative is called the Levi-Civita connection. In general, a "directional derivative" on a vector bundle is called a "connection".

(3) The above definitions of directional derivative are important even for a general vector bundle. For example if we want to prove that there is a nowhere vanishing section of a certain vector bundle, ideally, we would want to take a "constant" section. But to even define that, we need to know the notion of a directional derivative.

(4) The notion of "curvature" seems to depend on one derivative of the Christoffel symbol (or alternatively, two derivatives of the metric).

The above mentioned observations force us to define a connection $\nabla_W s$ on vector bundles. It is suppose to represent how fast a section s is changing along the tangent vector W. In fact, if W is a vector field, then $\nabla_W s$ better be a section of the vector bundle itself. So, we have

Definition 2.1. Suppose V is a smooth rank-r real vector bundle (a similar definition holds for complex vector bundles) over a smooth manifold M. Suppose $\Gamma(V)$ is the (infinite-dimensional) vector space of smooth sections of V over M. Suppose X is a vector field on M. Then a connection (sometimes called an affine connection) ∇ on V is a map $\nabla_X : \Gamma(V) \to \Gamma(V)$ satisfying the following properties.

(1) *Tensoriality in* X : If s is a smooth section of V, X_1, X_2 are two vector fields, and f_1, f_2 are two smooth functions, then $\nabla_{f_1 X_1 + f_2 X_2}(s) = f_1 \nabla_{X_1} s + f_2 \nabla_{X_2} s$. In other words, the value of $\nabla_X s$ at p depends only on the value of X at p but not on the derivatives of X.
(2) *Linearity in* s : If s_1, s_2 are two sections and c_1, c_2 are two real numbers, then $\nabla_X(c_1 s_1 + c_2 s_2) = c_1 \nabla_X s_1 + c_2 \nabla_X s_2$.
(3) *Leibniz rule* : If f is a smooth function and s is a section, $\nabla_X(fs) = f\nabla_X s + df(X)s = f\nabla_X + X(f)s$.

The first assumption (tensoriality in X) can be stated in another nice way : Suppose we fix s. Then the map $(X, \alpha) \to \alpha(\nabla_X s)$ is a map from $Vect\ fields \times \Gamma(V^*) \to C^\infty\ functions$ which is multilinear (over functions). It can be proved that there exists a smooth section $T_s \in \Gamma(T^*M \otimes V^{**} \simeq V)$ such that $T_s(X, \alpha) = \alpha(\nabla_X s)$.

Thus ∇ can be thought of as a map $\Gamma(V) \to \Gamma(V \otimes T^*M)$ given by $s \to \nabla s$. The space $\Gamma(V \otimes T^*M)$ is commonly called "vector-valued 1-forms". Moreover, in this framework, a connection satisfies $\nabla(fs) = df \otimes s + f\nabla s$.

Locally, suppose $e_1, \ldots, e_r$ is a frame (i.e. a collection of smooth local sections such that every point, they form a basis for the fibre) giving a local trivialisation of V. Then every smooth section is of the form, $s = s^\mu e_\mu$ where s^μ are smooth functions. Therefore,

$$\nabla(s^\mu e_\mu) = ds^\mu \otimes e_\mu + s^\mu \nabla e_\mu = ds^\mu \otimes e_\mu + s^\mu A^\nu_{\ \mu} \otimes e_\nu = (ds^\mu + A^\mu_{\ \nu} s^\nu) \otimes e_\mu \tag{2.2}$$

where $A^\mu_{\ \nu}$ is an $r \times r$ matrix consisting of 1-forms. Note that $\nabla_X s = X(s^\mu) e_\mu + A_{\ \nu}(X)^\mu s^\nu e_\mu$. Suppose we change our trivialisation to $\tilde{e}_1, \tilde{e}_2, \ldots$. Then of course the matrix of 1-forms A will change to $\tilde{A}$. Let us calculate this change. Suppose $\tilde{e}_\mu = g^\nu_{\ \mu} e_\nu$, i.e., $\tilde{e} = eg$ where g is an invertible smooth matrix-valued function. Then since $s = \tilde{s}^\mu \tilde{e}_\mu = s^\nu e_\nu$, we see that $\tilde{e}\vec{\tilde{s}} = eg\vec{\tilde{s}} = e\vec{s}$. Hence $\vec{\tilde{s}} = g^{-1}\vec{s}$. Since $\nabla_X s$ is a section, $\vec{\tilde{\nabla}}_X s = g^{-1}\vec{\nabla}_X s$, i.e., $\vec{\tilde{\nabla}} s = g^{-1}\vec{\nabla} s$. Hence,

$$\begin{aligned} d\vec{\tilde{s}} + \tilde{A}\vec{\tilde{s}} = g^{-1}(d\vec{s} + A\vec{s}) &\Rightarrow d(g^{-1}\vec{s}) + \tilde{A}g^{-1}\vec{s} = g^{-1}(d\vec{s} + A\vec{s}) \\ -g^{-1}dgg^{-1}\vec{s} + \tilde{A}g^{-1}\vec{s} = g^{-1}A\vec{s} &\Rightarrow \tilde{A} = g^{-1}Ag + g^{-1}dg \end{aligned} \tag{2.3}$$

In more familiar terms, rewriting $\vec{\tilde{s}} = g\vec{s}$ where g are the transition functions (i.e., replacing g^{-1} by g), we see that $\tilde{A} = gAg^{-1} - dgg^{-1}$.

So A does not change like a tensor. However, the cool thing is that, suppose ∇_1 is one connection. Then, if ∇_2 is any other connection, $(\nabla_2 - \nabla_1)(fs) = f(\nabla_2 - \nabla_1)s$. In other words, the difference of any two connections is an Endomorphism of the vector bundle. Locally, $\tilde{A}_2 - \tilde{A}_1 = g(A_2 - A_1)g^{-1}$. In other words, $A_2 - A_1$ is a section of $End(V) \otimes T^*M$. So the space of connections is an affine space (a vector space without a preferred choice of an origin).

We prove a useful little lemma here

Lemma 2.2. *If two smooth sections $s_1, s_2 : M \to V$ satisfy $s_1 = s_2$ on a neighbourhood U of p, then $\nabla s_1(p) = \nabla s_2(p)$. T(That is, the directional derivative at p depends only on local information about s near p.)*

Proof. Taking $s = s_1 - s_2$, we just have to show that $\nabla s(p) = 0$ if $s = 0$ on U. Indeed, taking a local trivialisation, $s = s^i e_i$ and hence $\nabla s(p) = (ds^i(p) + ([A](p)\vec{s}(p))^i)e_i = 0$. □

In fact, the above observation shows that in order to know ∇s at p, it is enough to take any smooth section $\tilde{s}$ on U, define $s = \rho\tilde{s}$ where ρ is a bump function vanishing outside U (and equal to 1 in a smaller neighbourhood of p), and find out $\nabla s(p)$ for all such sections.
Before going further, we prove a very very useful lemma. (This is like the existence of normal coordinates.)

Lemma 2.3. *Suppose ∇ is a connection on V. Suppose $p \in M$. There exists a trivialisation such that $A(p) = 0$ in this trivialisation.*

Proof. Choose any trivialisation in a neighbourhood U of p. Assume that (x, U) is also a coordinate chart for M such that p corresponds to $x = 0$. Let $\nabla = d + \tilde{A}$ on U. If we change the trivialisation using a transition function g, then $A = g\tilde{A}g^{-1} - dgg^{-1}$. Suppose $\tilde{A}(p) = B_i dx^i$ where B_i are real (or complex) $r \times r$ matrices. Define $g = I + x^i b_i$. For sufficiently small x, g is invertible. Now $g(p) = g^{-1}(p) = I$ and $dg = B_i dx^i = \tilde{A}(p)$. Thus $A(p) = \tilde{A}(p) - \tilde{A}(p) = 0$. □

Note that the trivial bundle $M \times \mathbb{R}^r$ has an obvious connection - the usual directional derivative. Indeed, there is a global trivialisation. Set $A = 0$ and define $\nabla s = d\vec{s}$.

Another point : If $T : V_1 \to V_2$ is a bundle isomorphism, and V_2 has a connection ∇, we can define a connection on V_1 : $(T^*\nabla)s = T^{-1}(\nabla T(s))$. This is called the pullback connection. Locally, the connection matrix of one-forms is $T^{-1}AT + T^{-1}dT$.

Theorem 2.4. *Every vector bundle has a connection.*

Proof. Suppose M is covered by a locally finite cover U_α of trivialisation open sets for V. Suppose $T_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times \mathbb{R}^r$ is the trivialising isomorphism of bundles. As discussed, there is an obvious connection ∇_α on the trivial bundle $U_\alpha \times \mathbb{R}^r$. Define $\tilde{\nabla}_\alpha = T_\alpha^* \nabla_\alpha$ as a connection on V on the set U_α, i.e., if s is any local section on U_α, $\tilde{\nabla}_\alpha s$ is a section of $T^*U_\alpha \times V|_{U_\alpha}$. Suppose ρ_α is a partition-of-unity subordinate to U_α.

Now, define $\nabla s = \sum_\alpha \rho_\alpha \tilde{\nabla}_\alpha s$. The meaning of this statement is "Take s, restrict it to U_α, calculate $\tilde{\nabla}_\alpha s$ as a section over U_α, multiply by ρ_α and extend it to all of M by 0 outside U_α, and sum over all α. It is a finite sum at every point because of local finiteness of the cover. Thus we have a section of $V \otimes T^*M$"

We still have to prove that ∇ is a connection. Indeed,

$$\begin{aligned} \nabla(fs) &= \sum_\alpha \rho_\alpha \tilde{\nabla}_\alpha(fs) = \sum_\alpha \rho_\alpha(df \otimes s + f\tilde{\nabla}_\alpha s) \\ &= df \otimes s \sum_\alpha \rho_\alpha + f\nabla s = df \otimes s + f\nabla s \end{aligned} \tag{2.4}$$

□

So every vector bundle can be equipped with a way to take directional derivatives. There can be more than one way (infinitely many in fact). We can now define the notion of a "constant", rather a "parallel" section.

Definition 2.5. A smooth section s is said to be parallel with respect to a connection ∇ if it satisfies $\nabla s = 0$.

We can do better. Suppose $\gamma : [0,1] \to M$ is a smooth curve. Assume that s_0 is a vector in $V_{\gamma(0)}$.

Definition 2.6. The parallel transport of s_0 is a section s on a neighbourhood of the image of γ such that $\nabla_{\gamma'(t)} s = 0$ on the image of γ (where we are assuming that $\gamma'(t)$ has been extended arbitrarily to a smooth vector field on a smaller open subset of the neighbourhood on which s is defined).

The neighbourhood does not make any difference in the above definition. The definition locally means this : If we choose a trivialisation and a coordinate chart on the manifold, we are required to solve an ODE : $\frac{d\vec{s}}{dt} + A_{\gamma(t)}(\gamma'(t))\vec{s} = 0$ with $\vec{s}(0) = \vec{s}_0$. Of course this system of ODE has a unique smooth solution for a short period of time. In fact, it can be proven to have a solution for all time.