NOTES FOR 3 MARCH (TUESDAY)

1. Recap

(1) Defined elliptic operators.
(2) Gave a wrong proof of the fact that elliptic regularity for H^s can be reduced to proving an inequality for smooth sections u. Instead, one can prove that if $u \in H^o$, the desired inequality holds and that if $s > o$, then the result follows from the $s = o$ case.

2. Elliptic operators - Regularity

Now we prove the aforementioned theorem (for smooth solutions as opposed to weak solutions). The full elliptic regularity result for weak L^2 solutions is more complicated (and can be found in Folland's book or in Kodaira's book). We shall give a sketch of the ideas later.

Proof. Writing $u = \sum \rho_\mu u$, we see that if we can prove that $\|\rho_\mu u\|_{H^{s+o}} \leq C_s(\|L(\rho_\mu u)\|_{H^s} + \|\rho_\mu u\|_{L^2})$ we will be done. Indeed (from now onwards all constants depending on s (and on the ellipticity constants and upper bounds on the coefficients) will be denoted by abuse of notation as C_s),

$$\|u\|_{H^{s+o}} \leq C_s \sum_\mu (\|L(\rho_\mu u)\|_{H^s} + \|\rho_\mu u\|_{L^2}) \leq \tilde{C}_s \sum_\mu (\|\rho_\mu Lu\|_{H^s} + \|u\|_{L^2}) + C_s \sum_\mu \|[L, \rho_\mu]u\|_{H^s}$$

$$\leq \tilde{C}_s(\|Lu\|_{H^s} + \|u\|_{L^2}) + C_s\|u\|_{H^{s+o-1}} \tag{2.1}$$

Using the interpolation inequality we see that $C_s\|u\|_{H^{s+o-1}} \leq \frac{1}{2}\|u\|_{H^{s+o}} + C\|u\|_{L^2}$. Thus we have reduced the problem to proving $\|\rho_\mu u\|_{H^{s+o}} \leq C_s(\|L(\rho_\mu u)\|_{H^s} + \|\rho_\mu u\|_{L^2})$.

Let $p_\mu \in U_\mu$ be a fixed collection of points. Suppose $\tilde{\rho}_\mu$ is a bump function equal to 1 on the support of ρ_μ and having support in U_μ, then if the cover U_μ is chosen to be fine enough so that the coefficients of L do not vary much from their values at p_μ (the size of this cover will of course depend on the ellipticity constants and an upper bound on the derivatives of the coefficients), then $\tilde{L}_\mu = \tilde{\rho}_\mu L + (1 - \tilde{\rho}_\mu)a_o(p_\mu)^I\partial_I$ can be thought of as a uniformly elliptic operator (with bounded ellipticity constants) acting on the torus with variable coefficients. Thus, we have reduced the problem to proving the estimate on a flat torus with the trivial vector bundle (but with variable coefficients).

The rough idea is to cover the torus with lots of open sets such that the operator is not far from a constant coefficient one on those sets, i.e., $L - L(p_\mu)$ is small. Then we know that $\|\rho_\mu u\|_{H^{s+o}} \leq C_s(\|L(p_\mu)\rho_\mu u\|_{H^s} + \|\rho_\mu u\|_{L^2}) \leq C_s(\|L\rho_\mu u\|_{H^s} + \|\rho_\mu u\|_{L^2}) + C_s\|(L - L(p, \mu))\rho_\mu u\|_{H^s}$. Note that a single C_s works independent of what p_μ are simply because C_s depends only on the ellipticity constants and upper bounds on the coefficients. If the last term is smaller than $\frac{1}{2}\|u\|_{H^{s+o}}$ (for instance), then we are done. But if we make the cover small, we risk making the other factor large. This is the problem.

Firstly, we claim that it is enough to prove the estimate for $s = 0$. Indeed, if this is done, then

$$\|\partial_i u\|_{H^o} \leq C(\|L(\partial_i u)\|_{L^2} + \|\partial_i u\|_{L^2}) \leq C(\partial_i(Lu)\|_{L^2} + \|[L, \partial_i]u\|_{L^2} + \|u\|_{H^o})$$

$$\Rightarrow \|u\|_{H^{o+1}} \leq C(\|Lu\|_{H^1} + \|u|_{L^2}) \tag{2.2}$$

Inductively we can prove this for a general s.

Suppose we choose a fine enough cover of the torus so that $\|(L-L(p_\mu))u\|_{L^2} \leq \frac{1}{2C_0}\|u\|_{H^0}$. Then of course $\|(\rho_\mu(L-L(p_\mu))u\|_{L^2} \leq \frac{1}{2C_0}\|u\|_{H^o}$ because $\rho_\mu \leq 1$. Fix such a cover and a partition-of-unity (we will not make it any finer than this). Therefore,

$$\frac{1}{2}\|u\|_{H^o} \leq \sum_\mu C_0(\|L\rho_\mu u\|_{L^2} + \|\rho_\mu u\|_{L^2}) + C_0 \sum_\mu \|[(L-L(p,\mu)),\rho_\mu]u\|_{H^s}$$
$$\leq C_0 \sum_\mu (\|L\rho_\mu u\|_{L^2} + \|\rho_\mu u\|_{L^2}) + C_1\|u\|_{H^{o-1}}$$

(2.3)
$$\leq C_0(\|Lu\|_{L^2} + \|u\|_{L^2}) + C_0\sum_\mu \|[L,\rho_\mu]u\|_{L^2} + C_1\|u\|_{H^{o-1}} \leq C_0(\|Lu\|_{L^2} + \|u\|_{L^2}) + C_2\|u\|_{H^{o-1}}$$

If we can prove that $\|u\|_{H^{o-1}} \leq \frac{1}{3C_2}\|u\|_{H^o} + C\|u\|_{L^2}$, we will be done. Indeed, this follows from the interpolation inequality for Sobolev spaces. □

Now we return to the full elliptic regularity result (which we wrongly claimed to have proved in the last class by reducing it to the previous theorem). The key ideas are

(1) Distributions : The space of distributions of order s $H^{-s}(M,E)$ is defined to be the metric space completion of L^2 under the norm $\|v\|_{H^{-s}} = \|F_v\| = \sup_{u\in H^s} \frac{|(u,v)_{L^2}|}{\|u\|_{H^s}}$. It is not hard to prove using functional analysis that $(H^s)^* \simeq H^{-s}$. Note that if $v \in L^2$, then $\|v\|_{-s} \leq \|v\|_{L^2}$. Also, if $v \in H^{-s}$, then $|v(u)| \leq \|v\|_{-s}\|u\|_s$. It is not hard to see that the isomorphism $(H^s)^* \simeq H^{-s}$ is an isometry and hence H^{-s} is a Hilbert space. Some easy functional analysis also allows one to conclude that given $G \in (H^{-s})^*$, there is a unique $u \in H^s$ such that $G(v) = (u,v)_{L^2}$. As in the case of a torus, we define the derivative of a distribution through "integration-by-parts". The Sobolev inclusion and Rellich compactness still hold.

In fact, we claim that $v \in H^{-s}$ if and only if $\rho_\mu v \in H^{-s}(S^1\times S^1 \ldots, \mathbb{R}^r)$ (where $\rho_\mu v(u) = v(\rho_\mu u)$) where U_μ is a trivialising coordinate cover and ρ_μ is a partition-of-unity). Moreover, the H^{-s} norm is equivalent to $\sum_\mu \|\rho_\mu v\|_{H^{-s}(S^1\times S^1 \ldots)}$. Furthermore, one can prove that if $u \in L^2$ is a distributional solution of $Lu = f$ where $f \in H^{-o}$, then $\|u\|_{L^2} \leq C_{-l}(\|f\|_{H^{-l}} + \|u\|_{H^{-l}})$. This will be part of a HW.

(2) Difference quotients : Let u be a vector-valued H^{-s} distribution on the torus and $0 < h \leq 1$. The difference quotient $\Delta_{h,e_i}u$ is a vector-valued H^{-s} distribution defined as $\Delta_{h,e_i}u(v) = u(\Delta_{-h}v) = u(\frac{v(x-he_i)-v(x)}{h})$ for all $v \in H^s$. Here is a beautiful result about difference quotients.

Theorem 2.1. *Let $s \in \mathbb{R}$. The following spaces are on the torus.*
(a) *If $|u\|_{H^{s+1}} \leq C$, then $\|\Delta_h u\|_{H^s} \leq C \ \forall\ 0 < h < 1$.*
(b) *Conversely, if $u \in H^s$ and $\|\Delta_h u\|_{H^s} \leq C \ \forall\ 0 < h < 1,\ e_i$, then $|u\|^2_{H^{s+1}} \leq nC^2 + \|u\|^2_{H^s}$.*
(c) *If $|\alpha| = l$ and $u \in H^{s+l}$, then $\|[a_\alpha(x)\partial^\alpha, \Delta_h]u\|_{H^s} \leq C\|u\|_{H^{s+l}} \ \forall\ 0 < h < 1$ where C depends only on the upper bounds on the C^{s+1} norms of the coefficient a_α.*

Proof. The Fourier coefficients of the distribution $\Delta_h u$ can be easily calculated to be $\hat{\Delta_h u}(k) = \frac{\hat{u}(k)e^{\sqrt{-1}k_i h} - \hat{u}(k)}{h}$.

(a)

$$\|\hat{\Delta_h u}\|_{H^s}^2 = \sum_k |\hat{u}(k)|^2(1+|k|^2)^s k_i^2 \frac{\sin^2(\frac{k_i h}{2})}{(\frac{k_i h}{2})^2} \le C^2 \tag{2.4}$$

(b) For each i, take $h \to 0$ and use Fatou's lemma to conclude that

$$C \ge \sum_k |\hat{u}(k)|^2(1+|k|^2)^s k_i^2 \tag{2.5}$$

We add over i to get the result.

(c) Let smooth functions $u_n \to u$ in H^{s+l},

$$\|[a_\alpha(x)\partial^\alpha, \Delta_h]u_n\|_{H^s} = \|\Delta_h a_\alpha \partial^\alpha u_n(x+h)\|_{H^s} \le C\|u_n(x)\|_{H^s} \tag{2.6}$$

Taking $n \to \infty$ we get the result.

□