NOTES FOR 4 FEB (TUESDAY)

1. Recap

(1) Defined connections, looked at them locally, proved that they form an affine space, and proved that they exist on every vector bundle.
(2) Defined the pullback of a connection.

2. Connections and curvature

Now we turn to another notion arising from a connection. What if we want to take the second derivative ? There is a nice way to do this using a connection, but let us return to that later. For now, let us be very naive. Note that ∇ takes sections to vector-valued 1-forms. What if we want to apply ∇ again ? Unfortunately, unless we have a way to differentiate 1-forms, there is no meaning to differentiating $\omega \otimes s$. But we actually do have a way to differentiate 1-forms using the exterior derivative d ! So, define the following map $d^{\nabla} : \Gamma(V \otimes T^*M) \rightarrow \Gamma(V \otimes \Omega^2(M))$ given by $d^{\nabla}(\omega \otimes s) = d\omega \otimes s + \omega \wedge \nabla s$ and extending it linearly. Of course, $d^{\nabla}(f\omega \otimes s) = df \wedge \omega \otimes s + f d^{\nabla}(\omega \otimes s)$. So indeed, tensoriality holds and hence the image of d^{∇} is a vector-valued 2-form. Actually, let's take this opportunity to define $d^{\nabla} : \Gamma(V \otimes \Omega^r M) \rightarrow \Gamma(V \otimes \Omega^{r+1} M)$ as $d^{\nabla}(s \otimes \omega) = \nabla s \wedge \omega + s \otimes d\omega$.

It is natural to ask whether $(d^{\nabla})^2 = 0$ on sections (i.e. vector-valued 0-forms). But this is not true ! Indeed, locally, $d^{\nabla} s = (d\vec{s} + A\vec{s})$. Thus $(d^{\nabla})^2 s = d(d\vec{s} + A\vec{s}) + A \wedge (d\vec{s} + A\vec{s}) = 0 + d(A\vec{s}) + A \wedge d\vec{s} + A \wedge A\vec{s} = dA\vec{s} - A \wedge d\vec{s} + A \wedge d\vec{s} + A \wedge A\vec{s} = (dA + A \wedge A)\vec{s} = F\vec{s}$ where F is locally a matrix of 2-forms called the curvature of (V, ∇). In other words, $(d^{\nabla})^2 s$ depends linearly on s and not on any derivative of it ! More curiously, if we calculate how F changes when we change the trivialisation, we see that $\tilde{F} = gFg^{-1}$. In other words, F is actually a section of $End(V) \otimes \Omega^2(M)$. (We can do this calculation more invariantly by proving tensoriality, i.e., $(d^{\nabla})^2(fs) = f(d^{\nabla})^2 s$.)

Definition 2.1. The curvature F of a connection ∇ is a section of $End(V) \otimes \Omega^2(M)$ defined as $Fs = (d^{\nabla})^2 s$. It locally has the formula, $F = dA + A \wedge A$.

If V is a line bundle, $A \wedge A = 0$ and $F = dA$ is a global closed 2-form (because $End(L)$ is a trivial bundle).

Here is an interesting observation :

Lemma 2.2. *If (L, ∇) is a (real or complex) line bundle, then its curvature F is a globally defined closed 2-form whose De Rham cohomology class is independent of the connection chosen.*

Proof. We already saw that F is a globally defined close 2-form. Suppose $\nabla_1, \nabla_2 = \nabla_1 + a$ are two connections where a is a section of $End(L) \otimes T^*M$. Noting that $End(L)$ is trivial, a is a globally defined 1-form. Now $F_2 = dA_2 = dA_1 + da = F_1 + da$. Therefore $[F_2] = [F_1]$. ☐

Real line bundles are actually quite straightforward to study. They are either orientable (and hence trivial) or non-orientable. In either case, $L \otimes L$ has transition functions $g_{\alpha\beta}^2 > 0$. Thus $L \otimes L$ is always a trivial real line bundle.

Complex line bundles are much more complicated and interesting. The De Rham cohomology class $[\frac{\sqrt{-1}}{2\pi} F]$ associated to a complex line bundle L is denoted as $c_1(L)$ and is called the first Chern class of L. (The presence of $\sqrt{-1}$ and 2π is technical. It is done so that whenever you integrate this cohomology class against a 1-dimensional submanifold, you get an integer as the answer.)

Given connections ∇^v, ∇^w on vector bundles V and W respectively, there exists natural connections on $V \oplus W$, V^* (for this you only need ∇^v), and $V \otimes W$-

(1) $V \oplus W : \nabla^{v \oplus w}(s \oplus t) = \nabla^v s \oplus \nabla^w t$. It is easy to verify that this satisfy all the definition of a connection. Locally, $A^{v \oplus w} = A^v \oplus A^w$ (a block diagonal matrix). Therefore, $F^{v \oplus w} = F^v \oplus F^w$.

(2) $V \otimes W : \nabla^{v \oplus w}(s \otimes t) = \nabla^v s \otimes t + s \otimes \nabla^w t$. Unfortunately, not every section of $V \otimes W$ is of the form $s \otimes t$. It is not obvious that it is even of the form $\sum c_{\alpha\beta} s_\alpha \otimes t_\beta$ where s_α, t_β are global sections.

However, given a p, it is easy to see that there exist global smooth sections s_α, t_β such that $\sum c_{\alpha\beta} s_\alpha \otimes t_\beta = s$ on a neighbourhood U of p. Define $\nabla^{v \oplus w}(\sum c_{\alpha\beta} s_\alpha \otimes t_\beta) = \sum c_{\alpha\beta}(\nabla^v s_\alpha \otimes t_\beta + s_\alpha \otimes \nabla^w t_\beta)$. We have to show that it is well-defined (independent of choices of s_α, t_β) and is genuinely a connection. This will be given as a homework problem.

Locally, $A^{v \otimes w} = A^v \otimes I + I \otimes A^w$ where we are using the Kronecker product of the matrices. Moreover, $F^{v \otimes w} = F^v \otimes I + I \otimes F^w$.

(3) V^* : Define ∇s^* to satisfy $d(s^*(t)) = (\nabla s^*)(t) + s^*(\nabla t)$ where t is any section of V and s^* a section of V^*. This is indeed a connection (easy to see). Locally, suppose e_i is a frame for V, then e^{i*} defined by $e^{i*}(e_j) = \delta^i_j$ is a frame for V^*. In this frame, $(A^*)^j_{\ i} = (\nabla e^{i*})(e_j) = d(e^{i*}(e_j)) - e^{i*}(\nabla e_j) = -A^i_{\ j}$. Thus $A^* = -A^T$. Therefore $F^* = -F^T$.

In the case where $V = L$ is a line bundle, the curvature satisfies $F^* = -F$. Therefore, for a complex line bundle $c_1(L^*) = -c_1(L)$.

(4) If $E = S \oplus Q$, then given a connection ∇ on E, we can define connections on S and Q. Indeed, $\nabla^S s = \pi_1 \circ \nabla s$ where π_1 is the projection to S. So, for example, since $V \otimes V = Alt \oplus Sym$, we see that, given a connection on V, we have a connection on the alternating tensors. (More generally, $V \otimes V \otimes V \ldots = Alt \oplus$ *other things including* $Sym(V)$.) Hence, if we are given a connection on TM, we have a connection on T^*M and hence on $\Omega^k(M)$ for all k.

As a consequence, given a connection on V, we have a naturally defined connection on $V \otimes V \otimes V \ldots$. If $V = L$ is a line bundle equipped with a connection ∇ with curvature F, then $L \otimes L \otimes \ldots$ has a connection whose curvature is kF. So for a complex line bundle, $c_1(L \otimes L \ldots) = kc_1(L)$. In fact, $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$.

Now we specialise further to more important connections.

Definition 2.3. Suppose h is a metric on V. Then a connection ∇ on V is said to be metric compatible with h if for any two sections s_1, s_2, $d(h(s_1, s_2)) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2)$.

It turns out that this is equivalent to saying that parallel transport preserves dot products. Locally, choosing an orthonormal frame $e_1, \ldots, e_r$, (i.e. a collection of r smooth local sections such that at every point there are orthonormal) we see that $d(h(e_i, e_j)) = 0 = h(\nabla e_i, e_j) + h(e_i, \nabla e_j) = h(A^k_{\ i} e_k, e_j) + h(e_i, A^k_{\ j} e_k) = A^j_{\ i} + A^i_{\ j}$. Therefore A is a skew-symmetric (skew-Hermitian in the complex case) matrix of 1-forms in a local orthonormal trivialisation. In that trivialisation, $F = dA + A \wedge A$ is a skew-symmetric (or skew-Hermitian in the complex case) matrix of 2-forms.

Note that the trivial connection $\nabla = d$ on a trivial bundle is compatible with the trivial metric.

Theorem 2.4. *On a vector bundle V equipped with a metric h (whether real or complex), there exists a metric compatible connection ∇.*

The proof of this theorem is very similar to the previous one (indeed, replace "trivialisation" with "orthonormal trivialisation" everywhere). Just as before, if we are given one metric compatible connection ∇_0, every other metric compatible connection equals $\nabla_0 + a$ where $a \in \Gamma(End(V) \otimes T^*M)$ is a skew-symmetric (or skew-Hermitian in the complex case) endomorphism-valued 1-form.

Note that suppose we are given a connection ∇^m on T^*M and ∇^v on V, then we have a connection $\nabla^{m\otimes v}$ on $T^*M \otimes V$. Therefore, we can define the second derivative of a section s of V as $\nabla^{m\otimes v}\nabla^v s$. Likewise, we can define higher order derivatives.

Now we define a PDE on a manifold.

Definition 2.5. Suppose V and W be smooth manifolds. Let $C(M,V), C(M,W)$ be the set of smooth maps from M to V, W respectively. A k^{th} order partial differential operator L is a map $L : C(M,V) \to C(M,W)$ such that locally it is of the form $Ls(x) = F(x, s, \partial s, \partial^2 s, \ldots, \partial^k s)$ where F is a smooth function. A PDE is an equation of the form $Lu = f$.
If V and W are vector bundles, then a linear partial differential operator is one that takes sections of V to sections of W, and satisfies $L(a_1 u_1 + a_2 u_2) = a_1 L(u_1) + a_2 L(u_2)$ where a_1, a_2 are constants.

Note that the above notion is well-defined. Indeed, if you change trivialisations and coordinates, you will get a different F but it will remain smooth and depend only on k derivatives of s.
We can finally come up with examples of PDE on manifolds :

(1) Any PDE in $\mathbb{R}^n$ does the job.
(2) More non-trivially, the Laplace equation $\Delta u = f$ on a torus is an example of a second-order linear PDE. A second order non-linear PDE on a torus is $\Delta u = e^u - f$. (If $f > 0$ this PDE turns out to have a unique smooth solution. Note that if $f = 1$, there is an obvious solution, i.e., $u = 0$.)
(3) $Lu = du = f$ where u is a k-form.
(4) $\nabla u = f$ where $u \in \Gamma(V)$ and $f \in \Gamma(V \otimes T^*M)$.
(5) $\nabla^{T^*M} du = f$ where u is a smooth function and f is a $(0,2)$-tensor. This equation is a second order linear PDE.
(6) A harmonic map $f : M \to N$ satisfies a second order nonlinear PDE (that we may write later).
(7) The Ricci flow is a second-order nonlinear PDE for the metric on a manifold.
(8) The Einstein equations in General Relativity are a second-order nonlinear PDE for a Lorentzian metric.
(9) The Navier-Stokes equation is a second-order nonlinear PDE.
(10) The Monge-Ampère equation is a second order nonlinear PDE.