NOTES FOR 6 FEB (THURSDAY)

1. Recap

(1) Defined curvature.
(2) Defined connections on various bundles arising from V and W.
(3) Defined PDE, linear PDE, and gave several examples of linear and nonlinear PDE.

2. Connections and curvature

We now come to the a very special metric-compatible connection on TM for a Riemannian manifold (M, g). This connection is determined completely by the metric.

Theorem 2.1. *Suppose (M, g) is a Riemannian manifold. There exists a unique metric compatible connection ∇ on TM such that it is torsion-free, i.e., for any two smooth vector fields X, Y,*

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{2.1}$$

This connection is called the Levi-Civita connection of the metric g. Commonly, its curvature is simply called the curvature of g.

Proof. We will do this in two ways :

(1) *Using coordinates* : Locally, ∇Y has components $(d + A)\vec{Y} = dY^i + A^i_{\ j} Y^j$ where A is an $m \times m$ matrix of 1-forms. So $A^i_{\ j} = \Gamma^i_{jk} dx^k$ where Γ^i_{jk} are a bunch of locally defined functions (the Christoffel symbols). So $\nabla_X Y$ is locally $\frac{\partial Y^i}{\partial x^j} X^j + \Gamma^i_{jk} X^k Y^j$. Take $X = \frac{\partial}{\partial x^a}$ and $Y = \frac{\partial}{\partial y^b}$ (suitably extended to all of M by a bump function). Now the torsion-free property implies that $\nabla_X Y - \nabla_Y X = 0$. In other words, $\Gamma^i_{ab} = \Gamma^i_{ba}$. In any normal coordinate system, at p, metric compatibility means that $A(p)$ is a skew-symmetric matrix, i.e.,

$$\Gamma^i_{ab}(p) = -\Gamma^a_{ib}(p) = -\Gamma^a_{bi}(p) = \Gamma^b_{ai}(p) = \Gamma^b_{ia}(p) = -\Gamma^i_{ba}(p) = -\Gamma^i_{ab}(p) \tag{2.2}$$

which means that $\Gamma^i_{ab}(p) = 0$. So if the LC connection exists, it is unique.

Define the Levi-Civita connection as :$\nabla Y_X(p) = \frac{\partial Y^i}{\partial x^j}(p) X^j(p)$ in any normal coordinate system at p. The fact that this is a connection is easy to see. (Linearly and tensoriality at p are obvious. The Leibniz rule at p is a consequence of the product rule for derivatives.)

(2) *Invariantly* :

$$\begin{aligned}
g(\nabla_X Y, Z) &= g([X, Y] + \nabla_Y X, Z) \\
&= g([X, Y], Z) + Y(X, Z) - g(X, \nabla_Y Z) = g([X, Y], Z) + Y(X, Z) - g(X, [Y, Z] + \nabla_Z Y) \\
&= g([X, Y], Z) + Y(X, Z) - g(X, [Y, Z]) - Z(g(X, Y)) + g(\nabla_Z X, Y) \\
&= g([X, Y], Z) + Y(X, Z) - g(X, [Y, Z]) - Z(g(X, Y)) + g([Z, X], Y) + g(\nabla_X Z, Y) \\
&= g([X, Y], Z) + Y(X, Z) - g(X, [Y, Z]) - Z(g(X, Y)) + g([Z, X], Y) + X(g(Z, Y)) - g(Z, \nabla_X Y)
\end{aligned}$$

$$\Rightarrow 2g(\nabla_X Y, Z) = g([X, Y], Z) + Y(X, Z) - g(X, [Y, Z]) - Z(g(X, Y)) + g([Z, X], Y) + X(g(Z, Y)) \tag{2.3}$$

This determines the connection completely (you can verify that this is indeed a connection) and is called Kozul's formula for the Levi-Civita connection.

□

Using the Kozul formula you can see that the Christoffel symbols have exactly the formula we wrote whilst studying geodesics. In fact, it is not hard to see that a geodesic is simply a curve γ such that $\nabla_{\gamma'}(\gamma') = 0$.

The torsion-free condition appears mysterious but there is a physics way of looking at it involving carrying rods along geodesics which start rotating in the presence of torsion. Indeed, consider the connection ∇ defined on $T\mathbb{R}^3$ as (suppose X, Y, Z are coordinate vector fields - example on mathoverflow),

$$\begin{aligned} \nabla_X Y = Z, \nabla_X Y = -Z \\ \nabla_X Z = -Y, \nabla_Z X = Y \\ \nabla_Y Z = X, \nabla_Z Y = -X. \end{aligned} \tag{2.4}$$

A body undergoing parallel translation for this connection spins like an American football: around the axis of motion with speed proportional to its velocity. So the geodesics are straight lines, and this connection preserves the standard metric, but it has torsion and is thus not the Levi-Civita connection.

Actually, there is another way of looking at the torsion-free condition.

Theorem 2.2. *Suppose M is a manifold. Let ∇^* be the induced connection on T^*M from any connection on TM. Then $d^{\nabla^*} : \Omega^1(M) \to \Omega^2(M)$.*

1. *$(d^{\nabla^*} - d)\omega$ satisfies tensoriality and hence there exists a tensor $T \in \Gamma(T^{**}M \simeq TM \otimes \Omega^2(M))$ such that $T_\omega(\_, \_) = (d^{\nabla^*} - d)(\omega)$.*
2. *$T(X, Y) = \omega(\nabla_X Y - \nabla_Y X - [X, Y])$. Thus for the Levi-Civita connection, $d^{\nabla^*} = d$.*

Proof.
1. $(d^{\nabla^*} - d)(f\omega) = df \wedge \omega + f d^{\nabla^*}\omega - df \wedge \omega - f d\omega = f(d^{\nabla^*} - d)\omega$. Hence, by tensoriality there exists such a tensor T (T is called the Torsion tensor of ∇).
2. Suppose

$$\begin{aligned} T_\omega(X, Y) = T_\omega(X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial x^j}) = X^i Y^j T(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = X^i Y^j (d^{\nabla^*} - d)(\omega)(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \\ = X^i Y^j (d^{\nabla^*} - d)(\omega_k dx^k)(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = X^i Y^j (A^*)^a_{\ k} \wedge \omega_a dx^k((\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})) \\ = X^i Y^j (\delta^k_j (A^*)^a_{\ k}(\frac{\partial}{\partial x^i})\omega_a - \delta^k_i (A^*)^a_{\ k}(\frac{\partial}{\partial x^j})\omega_a) = X^i Y^j (\omega_a \left(-A_j^{\ a}(\frac{\partial}{\partial x^i}) + A_i^{\ a}(\frac{\partial}{\partial x^j}) \right)) \\ = \omega(\nabla_X Y - \nabla_Y X - [X, Y]) \end{aligned} \tag{2.5}$$

□

As for the connection and curvature matrices for the Levi-Civita connection, $A^i_{\ j} = \Gamma^i_{j\mu} dx^\mu$ and $F^i_j = F^i_{j\mu\nu} dx^\mu \wedge dx^\nu$ which is a complicated expression involving two derivatives of the metric g. Typically, one writes R (standing for Riemann) for the curvature tensor instead of F. It satisfies various symmetries. It also satisfies the following expression (which is usually given as a definition).

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{2.6}$$

So the curvature matrix is $F^i_{\_j} = dx^i(R(,)\frac{\partial}{\partial x^j})$. While the curvature is important, it is too much information to keep track of. Here are other "curvatures" derived from this basic object that may sometimes be easier to handle.

(1) Sectional curvature of a two-plane spanned by an orthonormal set $X, Y \in T_pM$: This quantity is $g(R(X,Y)Y,X)$. It turns out that it is independent of the orthonormal frame chosen (using the symmetries of the Riemann tensor) and actually, using a polarisation-type identity, one can know the full Riemann tensor if one knows all sectional curvatures. Using this concept one can talk of "positively curved" (all sectional curvatures are positive everywhere) or "constant curvature" (all sectional curvatures are constant). For instance, we have two important theorems.
   (a) Complete Riemannian manifolds with constant sectional curvature are an isometric quotient of space forms : Hyperbolic space, or Euclidean space, or the Sphere. (Killing-Hopf theorem)
   (b) If the sectional curvature of a complete manifold is non-positive, then the universal cover is diffeomorphic to $\mathbb{R}^n$ (Cartan-Hadamard theorem).

(2) Ricci curvature : $Ricc(Y,Z) = tr(X \to R(X,Y)Z)$, i.e., $Ricc_{ab} = R^c_{\_bca}$. It turns out that $Ricc(X,Y) = Ricc(Y,X)$. This tensor (like the metric) is symmetric. In fact, one can prove that for $1,2,3$ dimensions, the Ricci tensor determines the full Riemann tensor. (Not beyond three dimensions though.) The Ricci curvature is very important. Here is a beautiful theorem (Bonnet-Myers) that illustrates its importance.

   Theorem 2.3. *If $k > 0$, $m = dim(M)$, and g is a complete Riemannian metric satisfying $Ric(p) - (m-1)kg(p) \geq 0 \forall\, p \in M$ (as semi-positive definite matrices), then $diam(M) \leq \frac{\pi}{\sqrt{k}}$. Hence M is compact. Moreover, since the universal cover is also compact (by pulling back the metric), the fundamental group is finite.*

(3) Scalar curvature : $S = Ricc_{ab}g^{ab}$. This curvature has the advantage (and disdvantage) of being a single number. The scalar curvature determines the full curvature in dimensions $1,2$. (Actually, all curvatures in dim 1 are zero.) The scalar curvature can be interpreted as follows :

$$\frac{Vol(B_\epsilon(p) \subset M}{B_\epsilon(0) \subset \mathbb{R}^m} = 1 - \frac{S}{6(m+2)}\epsilon^2 + O(\epsilon^4). \tag{2.7}$$

   Therefore, if $S > 0$, then balls in the manifold have smaller volume because they curve more.

   The Yamabe problem asks the following : On a compact manifold, given a metric g_0, is there is a function f so that $g = e^f g_0$ has constant scalar curvature ? (The answer is now known to be "yes"). If the "constant" of the scalar curvature is negative, then it is not incredibly hard to prove the theorem. There are obstructions to finding metrics of positive scalar curvature on manifolds (you can't always do it). Even if you can find one, the Yamabe problem in the positive case is very hard. Shockingly enough, its proof involves the positive mass theorem of general relativity.

3. Divergence, Stokes' theorem, and Laplacians

Suppose $u : M \to \mathbb{R}$ is a function on a Riemannian manifold (M,g) whose tangent bundle is equipped with the Levi-Civita connection. Then $\nabla u = \frac{\partial u}{\partial x^j} g^{ij} \frac{\partial}{\partial x^i}$ is called the gradient of u with respect to g. It is just dual to du using the metric g. Suppose c is a regular value of u, then $u^{-1}(c)$ is a submanifold of M of dimension $m-1$. The gradient ∇u is normal to this submanifold. Indeed, if

$\vec{v}$ is tangent to the submanifold, i.e., $\vec{v} = \frac{d\gamma}{dt}(0)$ where γ is a curve lying on $u^{-1}(c)$, i.e., $u(\gamma(t)) = c$, then $\frac{du}{dt} = 0$, i.e., $0 = \frac{\partial u}{\partial x^i}\frac{\partial \gamma^i}{\partial t} = (\nabla u)^j g_{ij}\frac{\partial \gamma^i}{\partial t}$. Thus ∇u is perpendicular to $\vec{v}$.

Suppose X is a smooth vector field. Define the divergence of X

Definition 3.1. $div(X) = \nabla_i X^i = \frac{\partial X^i}{\partial x^i} + \Gamma^i_{ik}X^k$. So in normal coordinates, it is the usual divergence at p. Note that $div(fX) = X(f) + f div(X)$.

Theorem 3.2. $\int_M div(X) vol_g = \int_{\partial M} i_X vol_g$ *where* $i_X\omega(Y_1, Y_2, \ldots) = \omega(X, Y_1, Y_2, \ldots)$. *If* $\vec{N}$ *is a unit outward pointing normal vector field on the boundary, then* $i_X vol_g = g(X, \vec{N}) dvol_{g|_{\partial M}}$.

Proof. Choose oriented normal coordinates x^i for g at $p \in M$. Now

$$div(X)(p)vol_g(p) = \sum_i \frac{\partial X^i}{\partial x^i}(p) dx^1 \wedge dx^2 \ldots dx^m(p) = d\Big(\sum_i X^i(-1)^{i-1}dx^1 \wedge \ldots dx^{i-1}\hat{dx^i} \wedge \ldots\Big)(p)$$

$$= d(i_X vol)(p) \tag{3.1}$$

Since the above equation is an equation of globally defined forms at p, it is independent of coordinates chosen. Thus $div(X)vol_g = d(i_X vol)$. By the usual Stokes' theorem, $\int_M div(X) vol_g = \int_{\partial M} i_X vol$. Now if $X = g(X, \vec{N})\vec{N} + Y$, then Y is tangent to the boundary. Choose oriented normal coordinates such that $x^1 = 0$ corresponds to the boundary (hence $\vec{N}(p) = \frac{\partial}{\partial x^1}$ and Y is a linear combination of ∂_i where $i \geq 2$)Then $i_X vol(p)|_{x^1=0} = g(X, \vec{N})(p) i_{\vec{N}(p)} dx^1 \wedge dx^2 \ldots (p) + i_{Y(p)} dx^1 \wedge dx^2 \ldots (p) = g(X, \vec{N}) vol_{g|_{\partial M}}(p)$. As before, this equation holds globally. $\square$

In particular, if M has no boundary, then $\int_M div(X) = 0$. Now define

Definition 3.3. The Laplacian Δu where u is a function on M is a function $\Delta u = div(\nabla u) = \frac{\partial}{\partial x^i}\left(g^{ij}\frac{\partial u}{\partial x^j}\right) + \Gamma^i_{ik}\frac{\partial u}{\partial x^j} g^{jk}$. So in normal coordinates, it is the usual Laplacian at p.

As an example, take the flat metric $g = d\theta^1 \otimes d\theta^1 + d\theta^2 \otimes d\theta^2 + \ldots$ on the torus. Then the Laplacian is easily seen to be the Laplacian we studied earlier. Here is an observation using Stokes :

$$\int_M \Delta u = \int div(grad(u)) dvol_g = 0 \tag{3.2}$$

So if $\Delta u = f$, a necessary condition is that $\int f dvol_g = 0$ (just like the torus). If $\Delta u = f$, then observe that for any smooth function v,

$$\int_M v\Delta u vol_g = \int_M v f vol_g \Rightarrow \int_M (div(v\nabla u) - \nabla v.\nabla u) vol_g = -\int_M \nabla v.\nabla u vol_g = \int_M u \Delta v vol_g \tag{3.3}$$

So we can define a distributional solution of $\Delta u = f$ as an L^2 function u such that the above holds for all smooth v.

What about the curl of a vector field X ? Firstly, given a vector field X, we can produce its dual 1-form $\omega_X(Y) = g(X, Y)$. We can then define $d\omega_X$ as a 2-form. If there is a way to take a 2-form α to an $m-2$ form $*\alpha$, then in 3-dimensions, $*\alpha$ will be a 1-form, whose dual is a vector field. This should be the curl. So we need a notion called the Hodge star $*$ taking k-forms to $m-k$ forms.

Definition 3.4. Given a k-form α on a compact oriented m-dimensional Riemannian manifold (M, g), $*\alpha$ is a $(m-k)$-form such that $\alpha \wedge *\beta = \langle \alpha, \beta \rangle_g vol_g$. Here the inner product on forms is defined as follows : Suppose at p, normal coordinates are chosen, i.e., $g_{ij}(p) = \delta_{ij}$, then $dx^{i_1}(p) \wedge dx^{i_2} \ldots \wedge dx^{i_k}(p)$ form an orthonormal basis at p for k-forms. Note that $vol(p) = dx^1(p) \wedge dx^2(p) \ldots dx^m(p)$.