NOTES FOR 7 JAN (TUESDAY)

1. Recap

(1) Proved that if f is smooth and periodic, then $u^{''} = f$ can be solved for smooth periodic function u if and only if $\int_0^{2\pi} f dx = 0$. Recast it in the language of linear algebra. (If $T : V \to V$, then $f \in Im(T)$ if and only if f is orthogonal to $Ker(T^*)$.) Also, if $f \in C^k$, then $u \in C^{k+2}$.

(2) Formally "proved" similar results using Fourier series.

(3) Wrote important theorems about Fourier series. The most important is that a smooth function has rapidly decaying Fourier coefficients (and the Fourier series converges uniformly). Conversely, if a_k is a rapidly decaying sequence, it corresponds to the Fourier coefficients of a smooth function.

2. Poisson ODE (cont'd)..

Note that the theorems above imply that if f is smooth and $\hat{f}(0) = 0$, then $\hat{u}(k) = -\frac{\hat{f}(k)}{k^2}$ (with $\hat{u}(0)$ being arbitrary) is also rapidly decaying (just as $\hat{f}(k)$ itself) and hence $\sum \hat{u}(k)e^{ikx}$ converges uniformly to a smooth function u such that $u^{''} = f$. Likewise, taking Fourier transform on both sides, any solution to this ODE is of the above form.

3. The Poisson equation on a torus

Now consider the equation $\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = f(x)$ where $f(x+2\pi \sum n_i e_i) = f(x)$ for a multiply-periodic function u. This can be thought of as a PDE on a torus $S^1 \times \ldots S^1$.

Except for the Hölder continuity criterion, all the other theorems mentioned above hold verbatim for Fourier series in higher dimensions, i.e., for $\hat{u}(\vec{k}) = \dfrac{1}{(2\pi)^n} \displaystyle\int\int\int \ldots f(\vec{x})e^{-i\vec{k}.\vec{x}} d\vec{x}$. The same proofs work.

If f is smooth and periodic, then its Fourier coefficients are rapidly decaying. In addition, if $\int\int \ldots f dx = 0$, then, $\hat{u}(\vec{k}) = -\frac{\hat{f}(\vec{k})}{|\vec{k}|^2}$ is rapidly decaying and hence $\sum_{\vec{k}} \hat{u}(\vec{k})e^{i\vec{k}.\vec{x}}$ (with $\hat{u}(0)$ being arbitrary) converges uniformly to $u(x)$ (and likewise for all its derivatives). Hence u is smooth, periodic, and satisfies the Poisson equation. By taking Fourier series on both sides, it is unique up to constants.

Another proof of uniqueness goes as follows : Suppose u_1, u_2 solve $\Delta u = f$. Then $\Delta(u_1 - u_2) = 0$. By multiplying by $u_1 - u_2$ and integrating over $[0, 2\pi] \times [0, 2\pi] \times \ldots$ and integrating-by-parts we get $-\displaystyle\int |\nabla(u_1 - u_2)|^2 = 0$. Hence $u_1 = u_2 + C$.

4. Weak solutions and Sobolev spaces

The strange thing about using Fourier series to solve the Poisson equation is that even if f is merely in L^2 and satisfies $\int\int f = 0$, the expression $\hat{u}(k) = \frac{\hat{f}(k)}{\vec{k}|^2}$ makes sense, is in l^2 and hence the corresponding Fourier series converges in L^2 to a function u. So even if u is not differentiable, it makes sense to talk about a "solution" to $\Delta u = f$! Such a "solution" is called a "weak solution". An important strategy to solve PDE is to first come up with such a weak solution (which can be done without too much difficulty using functional analysis or such "soft" techniques) and then prove that the weak solution is in fact, secretly smooth and satisfies the PDE honestly.

To gain more insight into this weak solution we found, suppose we are dealing with the ODE, and ϕ is a smooth periodic function, then by the Parseval-Plancherel theorem, $\int \phi^{''} u = \sum_k (-k^2)\hat{\phi}(k)\hat{u}(k) = \sum_k \hat{\phi}(k)\hat{f}(k) = \int \phi f$ for all such ϕ.

If u and f were actually smooth, then integration-by-parts implies that $\int \phi(u^{''} - f) = 0$ for all smooth ϕ. So choose $\phi = u^{''} - f$ to see that $u^{''} = f$. So indeed, for smooth solutions, this notion of a weak solution is equivalent to the usual notion (of a "strong solution). This argument can be easily generalised to higher dimensions.

The above arguments suggest that the following are important concepts

(1) **Weak derivative** : From now onwards, whenever we say "test function", we mean a smooth function with compact support. Suppose ϕ is a test function on a domain $U \subset \mathbb{R}^n$. If $u, v \in L^1_{loc}(U)$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a multiindex, then v is said to be the α^{th} weak derivative of u (written as $v = D^\alpha u$) provided

$$\int_U u D^\alpha \phi dx = (-1)^{\alpha_1+\alpha_2+\ldots} \int_U v\phi dx$$

for all test functions ϕ.

Before we prove that weak derivatives are unique, let us state the following useful lemma (we will do the proof a little later).

Lemma 4.1. *If $\int_U v\phi dx = 0$ for all test functions ϕ (where v is locally integrable), then $v = 0$ almost everywhere.*

Using this lemma we may prove the following theorem.

Theorem 4.2. *If $D^\alpha u$ exists weakly, then it is uniquely defined upto a set of measure zero.*

Proof. Suppose v_1, v_2 are two weak derivatives of u. Then $\displaystyle\int_U v_1\phi = \int_U v_2\phi$. Thus $\int_U (v_1 - v_2)\phi = 0$ for all test functions ϕ. This means that $v_1 - v_2 = 0$ almost everywhere. □

Here are examples and non-examples of weak differentiability.

(a) Let $U = (0, 2) \subset \mathbb{R}$. Suppose $u = x$ if $x \in (0, 1]$ and $u = 1$ if $1 \leq x < 2$. Define $v = 1$ if $0 < x \leq 1$and 0 if $1 < x < 2$. Then $u^{'} = v$ in the weak sense. Indeed, if ϕ is a test function, then $-\int_0^2 \phi^{'} u = -\int_0^1 \phi^{'} u - \int_1^2 \phi^{'} u = -(\phi u)(1-) + (\phi u)(0) + \int_0^1 \phi u^{'} - (\phi u)(2) + (\phi u)(1+) + \int_1^2 \phi u^{'} = \int_0^1 \phi v$.

(b) $U = (0, 2)$. Now $u(x) = x$ if $0 < x \leq 1$ and $u(x) = 2$ if $1 < x < 2$. We claim that u is not weakly differentiable. Suppose such a v exists. Then it is easy to see that on $(0, 1)$ $v = 1$ and on $(1, 2)$ it is equal to 0 (by uniqueness of weak derivatives). Then the above argument shows a contradiction because $u(1-) \neq u(1+)$.

Now we prove lemma 4.1. Before doing so, we need to take a detour into the concept of convolution and approximation. The slogan to keep in mind is "good convolved with bad is good" (something that Prof. Ravi Raghunathan of IIT Bombay taught me when I was an undergrad). From now onwards, U_ϵ is the set of all $x \in U$ whose distance from the boundary of U is at least ϵ.

Suppose ϕ is any smooth function on $\mathbb{R}$ with compact support centred around 0. Let

$\eta(x) = C\phi(|x|)$ where C is chosen so that $\int_{\mathbb{R}^n} \eta(x)dx = 1$. Define $\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta(x/\epsilon)$. Note that these functions are smooth, their integral is 1, and their supports are in $B(0,\epsilon)$. (They are supposed to be approximations of the Dirac delta.)

Suppose $f : U \to \mathbb{R}$ is locally integrable. Define f^ϵ on U_ϵ (its "mollification") to be $f^\epsilon = \eta_\epsilon * f = \int_U \eta_\epsilon(x-y)f(y)dy = \int_{B(0,\epsilon)} \eta_\epsilon(y)f(x-y)dy$. This operation is something like a weighted average of the values of f near x. So it "smooths out" f. The following are important properties of mollifiers.

(a) f^ϵ is smooth
(b) $f^\epsilon \to f$ a.e. as $\epsilon \to 0$.
(c) If $f \in C(U)$, then $f^\epsilon \to f$ uniformly on compact sets.
(d) If $f \in L^p_{loc}(U)$ then $f^\epsilon \to f$ in $L^p_{loc}(U)$ when $1 \leq p < \infty$.

Proofs :

(a) $f^\epsilon = \int_U \eta_\epsilon(x-y)f(y)dy$. If we can take the derivatives inside the integral sign, then indeed f^ϵ will be smooth. We can do so by the dominated convergence theorem. Indeed, $\lim_{h_1\to 0} f^\epsilon(x+(h_1,0,\ldots))-f^\epsilon(x) = \lim_{h_1\to 0}\int_U \frac{\eta_\epsilon(x+(h_1,0,0..))-y)-\eta_\epsilon(x-y)}{h}f(y)dy.$ If we choose any sequence $h_{1n} \to 0$, then since $|\frac{\eta_\epsilon(x+(h_1,0,0..))-y)-\eta_\epsilon(x-y)}{h}| \leq C$ (by the mean value theorem), we see by DCT that the limit can be taken inside the integral. This argument proves that all the partial derivatives exist. By DCT we can show that these partials are also continuous. Continuing inductively, this shows that f^ϵ is smooth.

(b) The key trick behind all these convergence proofs in mollification is this : $f(x) = \int_{B(0,\epsilon)} \eta_\epsilon(y)f(x)dy$. So

$$|f^\epsilon(x) - f(x)| = |\int_{B_{0,\epsilon}} \eta_\epsilon(y)(f(x-y)-f(x))dy| \leq \int_{B_{0,\epsilon}} |\eta_\epsilon(y)(f(x-y)-f(x))|dy$$

Now $|\eta_\epsilon(y)| \leq \frac{C}{\epsilon^n}$. Moreover, $vol(B_\epsilon) = C\epsilon^n$. Therefore,

$$|f^\epsilon(x) - f(x)| \leq C\frac{\int_{B(0,\epsilon)} |f(x-y)-f(x)|dy}{vol(B(0,\epsilon))} = C\frac{\int_{B(x,\epsilon)} |f(z)-f(x)|dz}{vol(B(x,\epsilon))}$$

. By the Lebesgue differentiation theorem, the right hand side goes to 0 almost everywhere as $\epsilon \to 0$. This so-called Lebesgue differentiation theorem holds even in L^p (i.e. as $\epsilon \to 0$, if $f \in L^p_{loc}(U)$ $(1 \leq p < \infty)$, then $\frac{\int_{B(x,\epsilon)} |f(z)-f(x)|^p dz}{vol(B(x,\epsilon))} \to 0$ a.e. in x). This is a generalisation of the fundamental theorem of calculus.