Lecture 2 during Covid

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IISc

- Completed the proof of parabolic existence.

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- Proved the Riemannian uniformisation theorem using the method of continuity.

The uniformisation theorem for Genus ≥ 2 : Variational method

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- A power series expansion, the above inequality, and Poincaré 's inequality show the Moser-Trudinger inequality : $\int_M e^{\beta u^2} dA \leq \gamma$ for some positive β, γ and all $||u||_{H^1} \leq 1$ and $\fint u = 0$.

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- Therefore, the constraint is met by f above.

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- Putting the above together, we see that $E(f)$ is bounded below (in a coercive manner). Indeed, $E(f) = \frac{1}{2}\int_M |\nabla(f - \fint f)|^2 - \int_M K_0(f - \fint f) + 2\pi|\chi(M)| \fint f \geq -C + \frac{1}{C}||f - \fint f||_{H^1}^2$.

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- Since $f_n \to f$ strongly in L^2 and $||f||_{H^1} \leq \liminf ||f_n||_{H^1}$,

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- Since $f_n \to f$ strongly in L^2 and $||f||_{H^1} \leq \liminf ||f_n||_{H^1}$, we see that $||\nabla f||_{L^2} \leq \liminf ||\nabla f_n||_{L^2}$.

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- Since $f_n \to f$ strongly in L^2 and $||f||_{H^1} \leq \liminf ||f_n||_{H^1}$, we see that $||\nabla f||_{L^2} \leq \liminf ||\nabla f_n||_{L^2}$.
- Hence, $E(f) = \inf(E)$ and $E(u) = \inf E$ where $f = \fint f + u$.

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- Now we need to show that f is smooth.

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