# Lecture 3 during Covid

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IISc

- Proved the Riemannian uniformisation theorem for genus $\geq 2$ using a variational method.

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- By choosing large and small constants, we can trivially find $f_\pm$.

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- $\Delta f_i - \epsilon f_i = T(f_{i-1}) \leq T(f_-) \leq \Delta f_- - \epsilon f_-$ and hence $f_i \geq f_-$. Likewise, $f_i \leq f_+$.

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- By elliptic regularity and bootstrapping, $f$ is smooth and hence the solution we are looking for.