COVID LECTURE 1

1. Recap

(1) Sketched a proof of parabolic existence.
(2) Motivated the Riemannian uniformisation theorem.

Theorem 1.1. *Every parabolic equation has a unique smooth solution for all time, i.e., on* $[0,\infty)\times M$.

Proof. First we prove uniqueness. Indeed, if there are two solutions, then let $v = u_1 - u_2$. It satisfies $\frac{dv}{dt} = -Lv, v(0) = 0$. Now,

$$\frac{d(v,v)_{L^2}}{dt} = -2(Lv,v) \leq -\delta(v,v)_{L^2}. \tag{1.1}$$

Hence,

$$(v,v)(t) \leq (v,v)(0)e^{-\delta t}.$$

Thus $v \equiv 0$. The estimate on v (an "Energy estimate") is useful in its own right. One can similarly prove that if $\frac{dv}{dt} = -Lv + f$, then $(v,v)(t) \leq C(1+t)$.

Now we prove existence. Let e_n be a countable family of smooth eigenvectors with eigenvalues $\lambda_n > 0$ of L spanning L^2. Thus, $u_0 = \sum_n c_n e_n$ for any $u_0 \in L^2$ (and $f = \sum_n f_n e_n$). Since $u_0 \in L^2$, we see that $\sum_n |c_n|^2 < \infty$. First we prove that the quantity $\|u_0\|_k^2 = \sum_n |c_n|^2(1+\lambda_n)^{2k}$ is equivalent to the $H^{k2\theta}$ norm. Indeed, if $\|u_0\|_k < \infty$, then $(u_0, L^k e_n)_{L^2} = \lambda_n^k c_n$. If ϕ is a smooth section, then $\phi = \sum_n \phi_n e_n$. Thus, $L^k\phi \in L^2$ satisfies $(L^k\phi, e_n) = \phi_n \lambda_n^k$. Therefore, $(u_0, L^k\phi) = \sum_n c_n \lambda_n^k \phi_n$ and hence $L^k u_0 = f_k$ in the sense of distributions where $f_k \in L^2$. Therefore, $u_0 \in H^{k\theta}$ and $\|u_0\|_{H^{k2\theta}} \leq C_k \|u_0\|_k$. Conversely, if $u_0 \in H^{2k\theta}$, then $\|L^k u_0\|_{L^2} \leq C\|u_0\|_{H^{2k\theta}} < \infty$. Thus, $(L^k u_0, e_n) = (u_0, L^k e_n) = \lambda_n^k c_n$. Therefore, $\|u_0\|_k < \infty$ and $\|u_0\|_k^2 \leq \tilde{C}_k \|u_0\|_{H^{2k\theta}}^2$.

Define the function $u(t) = \sum_n c_n e^{-\lambda_n t} e_n + \frac{f_n}{\lambda_n}(1 - e^{-\lambda_n t})e_n$. Clearly $u(t) \in L^2$. Moreover, $\|u(t) - u_0\|_{L^2}^2 = \sum_n |c_n|^2(1-e^{-\lambda_n t})^2$ which by DCT converges to 0 as $t \to 0^+$.

Now we proceed to prove that $u(t,x)$ is C^∞ in x for every fixed $t \geq 0$ and that we can differentiate w.r.t x term-by-term. Since $\sum_n c_n e_n$ and $\sum_n f_n e_n$ are smooth (by assumption), their $\|.\|_k$ norms are finite for all (by the equivalence of norms above). Therefore, $\|u\|_{H^{2k\theta}} \leq C_k \ \forall \ k$. Hence u is smooth in x for all fixed $t \geq 0$. Moreover, by Sobolev embedding, the partial sum $s_N(t) = \|\sum_{n=1}^N u_n e_n\|_{C^{k,\alpha}} \leq \tilde{C}_k$ independent of N. Therefore, by Arzela-Ascoli, every subsequence has a subsequence that converges in C^l and in fact the limits are all $u(t)$ because $s_N(t) \to u(t)$ in L^2. Therefore, $u(t) \in C^l$ for all l and the term-by-term derivatives in x converge.

Now note that if $u(t) = \sum_n u_n(t)e_n$ where $\|s_N(t)\|_{H^k} \leq C_k$ independent of $N, t \geq 0$, then $\|s_N(t) - s_N(t_0)\|_k^2 \leq \sum_{n=1}^N (1+\lambda_n)^{2k}(2\lambda_n^2|c_n|^2 + \lambda_n^2|f_n|^2)|t-t_0|^2 \leq C_k$ and hence $\|s_N(t) - s_N(t_0)\|_{C^0} < \epsilon$ for $t - t_0$ small (if $t_0 = 0$, then $t \geq 0$). So $u(t,x)$is continuous in (t,x). Actually, this argument shows that $\partial_x^l u(t,x)$ is continuous too.

Likewise, $\|s'_N(t) - s'_N(t_0)\|_{C^0} < \epsilon$ for t close to t_0. So the term-by-term derivatives $s'_N(t)$ converge uniformly to a continuous function $v(t,x)$. Note that $\int_0^t v(a)da = \lim_{N\to\infty} \int_0^t s'_N(a)da = u(s)$ and hence by the FTC, $u'(t,x) = v(t,x)$ and moreover, $u'(t,x)$ is continuous in t,x. (Actually it shows that all the partials in x are also continuous.) Inductively, we can prove that u is smooth on $[0,\infty)\times M$

and that we can differentiate term-by-term.

Finally, an easy calculation shows that u satisfies the equation with the boundary conditions. □

2. Uniformisation theorem

Writing the resulting PDE down, we get (See list of formulas in Riemannian geometry on wikipedia to get the correct formula),

$$\Delta f = Ke^{-f} - K_0, \tag{2.1}$$

where K_0 is the Gaussian curvature of g_0, K is the new curvature, Δf is locally, at a point where we choose coordinates such that $g_0(p)$ is the Euclidean metric up to the second order, $\Delta f(p) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$. The question is - Can we solve this equation ? If so, is the solution unique ? The answer (which is supposedly blowing in the wind) is provided by the Riemannian uniformisation theorem -

Theorem 2.1. *In every conformal class of metrics $[g]$ on a compact oriented surface, there exists a unique (up to rescalings by positive constants) metric of constant curvature.*

Proof. It is actually quite hard to prove (shockingly enough) this theorem for genus $g = 0$, i.e., for a sphere ! (Of course there is one metric of positive constant curvature that even children (who do not believe the flat-earth theory) know about. The issue is that are there other conformal classes ? (There aren't) If there are, how do you prove that they have such metrics ? The technique we are going to describe below will run into serious challenges for $g = 0$.) In fact, this is no coincidence. It turns out that one generalisation of this observation has been proven recently by Chen-Donaldson-Sun (and apparently independently by Tian). It is called the Yau-Tian-Donaldson conjecture. Another generalisation called the Yamabe problem was solved earlier.

Let us take the next case of $g = 1$. Note that by the Gauss-Bonnet theorem, $\int K dA = \int_M K_0 dA_0 = 2\pi(2-2g) = 0$. Therefore we want $K = 0$. This means we have to solve

$$\Delta f = -K_0.$$

Let's prove uniqueness first. Indeed, if f_1, f_2 are two solutions, then $\Delta(f_1 - f_2) = 0$. Multiplying by $f_1 - f_2$ and integrating-by-parts we get

$$-\int_M |\nabla(f_1 - f_2)|^2 = 0.$$

Therefore $f_1 - f_2$ is a constant. The point of this calculation is "If you want to prove that the kernel of some operator is trivial, multiply by something and integrate-by-parts". Since the Laplacian is elliptic, the Fredholm alternative shows that we are done for $g = 1$.

For higher genus, we want $K < 0$. Now we are faced with a nonlinear PDE. Here is a beautiful method (originally due to Bernstein) to handle such PDE. It is called the method of continuity. Consider the following family of PDE indexed by a number $0 \leq t \leq 1$.

$$\Delta f_t = -e^{-f_t} - tK_0 + (1-t). \tag{2.2}$$

At $t = 0$ there is obviously a solution $\phi_0 = 0$. If we prove that the set of t for which there exists a smooth solution is both open and closed, then by connectedness, the set is $[0, 1]$.

(1) Openness : Basically, given a solution at $t = t_0$, we need to prove that there are solutions nearby. Consider the following map,

$$T(t, f) = \Delta f + e^{-f} + tK_0 - (1-t). \tag{2.3}$$

Naively speaking, if this was a map between finite dimensional things, then by implicit function theorem, if its derivative with respect to f is surjective then we will be done. Indeed, there is an implicit function theorem on Banach spaces.

Theorem 2.2. *Suppose X, Y, Z are Banach spaces, $C \subset X \times Y$ is open, and $f : C \to Z$ is C^1. Suppose $(a, b) \in C$ and $v \to Df_{(a,b)}(0, v)$ is a Banach space isomorphism from Y onto Z. Then locally, $z = f(x, y)$ can be solved for to yield a C^1 function g such that $y = g(x, z)$.*

Remark 2.3. In fact, there is one for Banach manifolds, i.e., Hausdorff topological spaces equipped with a maximal atlas consisting of open sets isomorphic to a open subsets of Banach spaces with the transition functions being smooth, i.e., the Fréchet derivatives (in the usual sense) exist as multilinear bounded maps. Typically they are required to be separable and metrisable. In fact a theorem of Henderson states that such beasts are diffeomorphic to open subsets of the separable Hilbert space.

Remark 2.4. The theorem follows from the inverse function theorem on Banach spaces, whose proof is exactly word-to-word the same as the finite-dimensional one. (The contraction mapping principle.) You may look at Lang's book for it.

The appropriate Banach spaces to consider are $\mathbb{R} \times C^{k+2,\alpha}$ and $C^{k,\alpha}$ for a given integers $k \geq 0$ and $\alpha > 0$. Why the α? (Hölder space) It is for a technical reason as we shall see in a moment. The "derivative" with respect to f being surjective is the same as saying, for every $v \in C^{k,\alpha}$ there exists a $u \in C^{k+2,\alpha}$ such that

$$\begin{aligned} &\frac{d}{ds}|_{s=0} T(t_0, f_{t_0} + su) = v \\ &\Rightarrow \Delta u - e^{-f_{t_0}} u = v. \end{aligned} \tag{2.4}$$

Borrowing from our intuition from linear algebra (it is easy to verify that the above equation for u is self-adjoint), i.e., using the Fredholm alternative, we simply need to show that the kernel is trivial. Indeed if u is in the kernel, then

$$\begin{aligned} \Delta u &= e^{-f_{t_0}} u \\ \Rightarrow -\int_M |\nabla u|^2 dA_0 &= \int_M u^2 e^{-f_{t_0}} dA_0. \end{aligned}$$

which means that $u = 0$.

(2) Closedness : This is usually the harder part of any method of continuity. What does it mean for the set to be closed ? It means that for any sequence $t_n \to t$ such that f_{t_n} exist, there exists a solution f_t at t. In other words, if we can prove that a subsequence $f_{t_{n_k}} \to f_t$ in $C^{2,\alpha}$ then we will be done. Beautifully enough, the Arzela-Ascoli theorem implies that (do this as an exercise) if $\beta > \alpha$ and a sequence w_n is bounded independent of n in $C^{2,\beta}$, then a subsequence converges in $C^{2,\alpha}$! Thus, to show closedness, it is enough to prove that solutions to equation 2.2 have a uniform $C^{2,\beta}$ estimate independent of t.

Indeed, such estimates are proven by improving upon lower order estimates -

Let's see if we can at least prove that $\|f_t\|_{C^0} \leq C$. Indeed, at the maximum of f_t, easy calculus shows that $\Delta f_t \leq 0$. (Second derivative test.) Therefore $-e^{-f_t(max)} - tK_0 + (1-t) \leq$

0. This means that $f_t(max) \leq C$. Likewise $f_t(min) \geq c$.

Actually, now we have some standard results in PDE theory (read Kazdan's notes for instance) that say effectively the following : If the right hand side of $\Delta f = h$ is bounded in L^p for all large p, then f is actually bounded in $C^{1,\alpha}$ for some $\alpha > 0$ (It is basically L^p regularity + Sobolev embedding). There is another result (Schauder's estimates) that implies that if the right hand side of $\Delta f = h$ is bounded in $C^{0,\alpha}$ and $\|f\|_{C^0} \leq C$, then actually $\|f\|_{C^{2,\alpha}} \leq C$. So combining all of these, we get our desired estimates. (These are called "a priori" estimates.)

As for uniqueness, suppose f_1, f_2 satisfy the equation for $K < 0$. Then

$$\Delta(f_1 - f_2) = K(e^{-f_1} - e^{-f_2})$$

$$\Rightarrow -\int_M |\nabla(f_1 - f_2)|^2 = K \int_M (f_1 - f_2)(e^{-f_1} - e^{-f_2}). \tag{2.5}$$

This means that $f_1 - f_2$ is a constant.
Actually, uniqueness is quite easy for all three cases $K = 0, > 0, < 0$ assuming the Killing-Hopf theorem of the next section.

By the way, for $K > 0$, here is a way to prove some things : Firstly, in the conformal class of the usual round metric, there exists a constant curvature metric (the round one). Then assuming one knows complex geometry one proves that there is only one complex structure on the sphere. (This involves a little bit of algebraic geometry.) Thus there is only one conformal class and we are done.

$\square$