HW 2 (due on 13th Feb (Thursday) in the class)

Please write your answers clearly and rigorously. Write your name in plain lettering (as opposed to cursive) and also staple all the pages.

1. Suppose (M, g) is a Riemannian manifold. Assume that $\psi_0 : M \to \mathbb{R}$ is a continuous Lipschitz function (with Lipcshitz constant $1/4$) which is differentiable almost everywhere such that $|d\psi_0|_g \leq \frac{1}{4}$ almost everywhere. Prove that there exists a smooth function $\psi : M \to \mathbb{R}$ such that $|\psi_0 - \psi| \leq 1$ and $|d\psi|_g \leq 1$.

2. Suppose $s \geq 0$. Define $H^{-s}(S^1 \times S^1 \ldots)$ as the collection of all complex sequences $a_{\vec{k}}$ where $\vec{k} \in \mathbb{Z}^n$ such that $\|a\|^2_{H^{-s}} = \displaystyle\sum_{\vec{k} \in \mathbb{Z}^n} |a_{\vec{k}}|^2 (1 + |k|^2)^{-s} < \infty$. (In other words, a is allowed to grow polynomially in some sense.)

   (a) Prove that if $s > l$ then $H^{-l} \subset H^{-s}$.

   (b) Prove that the map $T_a(b) = \displaystyle\sum_{\vec{k}} \bar{a}_{\vec{k}} b_{\vec{k}}$ is well-defined when $b \in H^s$ and $a \in H^{-s}$. Also proved that T_a is a bounded linear functional on H^s with $\|T_a\| = \|a\|_{H^{-s}}$.

   (c) Prove that there is a function $f_a \in H^s$ such that $T_a(b) = \langle b(x), f(x) \rangle_{H^s}$.

   (d) Prove that $a \to T_a$ is an isomorphism between H^{-s} and $(H^s)^*$. (H^{-s} is called the space of distributions of order s.)

   (e) Suppose $a \in H^{-s}$ satisfies $\sum |a_k|^2 (1 + |k|^2)^l < \infty$ where $l \geq 0$, then prove that $a_k = \hat{f}(\vec{k})$ where $f \in H^l$.

   (f) Prove that smooth functions are dense in the space of distributions of order s.

   (g) Define the notion of a derivative of a distribution as a distribution of a higher order. Also make the following precise : "If a sequence of distributions converge, then so do their derivatives."

   (h) Suppose $L : H^{s+l} \to H^s$ is an elliptic operator on the torus with constant coefficients, prove that $u \in ker(L^* : (H^s)^* \simeq H^{-s} \to (H^{s+l})^*) \simeq Coker(L)$ if and only u corresponds to a smooth solution to $L^*_{form} u = 0$.

3. Suppose (V, ∇^v), (W, ∇^w) are two bundles with connections on a smooth manifold M. Let $\Gamma(V)$, $\Gamma(W)$, $\Gamma(V \otimes W)$ be the spaces of smooth sections. Let $S \subset \Gamma(V \otimes W)$ be the subspace consisting of sections of the form $\displaystyle\sum_{i=1,j}^{N} c_{ij} s_i \otimes t_j$ where $s_i \in \Gamma(V), t_j \in \Gamma(W)$ and N is arbitrary, i.e., S consists of finite linear combinations of decomposable sections. Let X be a smooth vector field.

(a) Define $\nabla_X^{v\otimes w} \sum_{i=1,j}^{N} c_{ij} s_i \otimes t_j = \sum_{i=1,j}^{N} c_{ij} (\nabla_X^v s_i \otimes t_j + s_i \otimes \nabla_X^w t_j)$. Prove that it is linear on S, tensorial in X, and satisfies the Leibniz rule.

(b) Prove that if $\sum_{i=1,j}^{N} c_{ij} s_i \otimes t_j = \sum_{\alpha=1,\beta=1}^{M} \tilde{c}_{\alpha\beta} \tilde{s}_\alpha \otimes \tilde{t}_\beta$ on a neighbourhood U_p of a point p, then $\nabla_X^{v\otimes w}(\sum_{i=1,j}^{N} c_{ij} s_i \otimes t_j) = \nabla_X^{v\otimes w} \sum_{\alpha=1,\beta=1}^{M} \tilde{c}_{\alpha\beta} \tilde{s}_\alpha \otimes \tilde{t}_\beta$. (Hence $\nabla^{v\otimes w}$ is well-defined on S.)

(c) Given a point $p \in M$, prove that if $s \in \Gamma(V \otimes W)$, then there exists a section $\tilde{s} \in S$ such that $s = \tilde{s}$ on a neighbourhood U_p of p. Conclude that $\nabla^{v\otimes w}$ is a connection on $V \otimes W$.

Comment : The same proof (almost word-to-word) shows that $d^\nabla : \Gamma(\Omega^r(M) \otimes V) \to \Gamma(\Omega^{r+1}(M) \otimes V)$ is well-defined.

Optional question : Suppose M is compact. Show that $S = \Gamma(V \otimes W)$.