HW 4 and 5 (due on 26 March (Thursday) in the class)

Please write your answers clearly and rigorously. Write your name in plain lettering (as opposed to cursive) and also staple all the pages.

1. Suppose E is a vector bundle on a compact manifold M. Cover M with finitely many coordinate trivialising neighbourhoods U_μ such that $\bar{U}_\mu$ is still contained in a coordinate trivialising neighbourhood. Let u be a section of E and $u = u^i_\mu e_{i,\mu}$ be the local representation on U_μ. Define $C^{k,\alpha}$ as the space of C^k sections u such that $u_\mu \in C^{k,\alpha}(\bar{U}_\mu)$ and the norm $\|u\|_{C^{k,\alpha}} = \sum_\mu \|u_\mu\|_{C^{k,\alpha}}$. Prove that this is a Banach space and that if you make different choices of U_μ and the trivialisations and coordinates, you get a quasi-isometric Banach space.

2. Prove that

   (a) A section v of E is in H^{-s} (where $s \geq 0$ is an integer) if and only if $\rho_\mu v \in H^{-s}(S^1 \times S^1 \ldots, \mathbb{R}^r)$.

   (b) Prove that the H^{-s} norm is equivalent to $\sum_\mu \|\rho_\mu v\|_{H^{-s}(S^1 \times S^1 \ldots)}$ where ρ_μ is a partition-of-unity.

   (c) For a constant coefficient elliptic operator of order l on the torus, prove that if $u \in H^t$ (where $t \in \mathbb{R}$) is a distributional solution of $Lu = f$ and $f \in H^s$, then $u \in H^{s+l}$ and that there is a constant C_s depending only on the ellipticity constants, the metrics, connections, and upper bounds on norms of the coefficients (which norms ? up to you to figure out) such that $\|u\|_{H^{s+l}} \leq C_s(\|u\|_{H^{s+l-1}} + \|f\|_{H^s})$ and if $s > 0$, then one can replace $\|u\|_{H^{s+l-1}}$ with $\|u\|_{L^2}$).

   (d) Let L be a variable coefficient elliptic operator of order l on the torus and p be a point on the torus. Then prove that given a constant $K > 0$, a smooth function ρ, there exists an open neighbourhood whose size can be bounded below by a constant depending only on the ellipticity constants, K, s, an upper bound on a norm of ρ and upper bounds on the norms of the coefficients of L such that for every $u \in L^2$, $\|\rho(L - L(p))u\|_{H^{-l}} \leq K\|u\|_{L^2}$. (Hint : Use the Young Convolution Inequality.)

   (e) Let L be a variable coefficient elliptic operator of order l on the torus. Prove that there is a constant C_{-l} such that if $u \in L^2$ is a distributional solution of $Lu = f$ where $f \in H^{-l}$, then $\|u\|_{L^2} \leq C_{-l}(\|f\|_{H^{-l}} + \|u\|_{H^{-l}})$.

(f) Let L be an elliptic operator on a compact manifold M. If $u \in L^2(E)$ is a distributional solution of $Lu = f$ where $f \in H^s$ ($s \geq 0$ is an integer), then prove that $u \in H^{s+l}$ by reducing it to a variable coefficient elliptic operator on the torus. Conclude that if f is smooth, then so is u.