HW 6

Please write your answers clearly and rigorously. Write your name in plain lettering (as opposed to cursive) and also staple all the pages.

1. Prove the Moser-Trudinger inequality $\int e^{\beta u^2} \leq C$ if $\|u\|_{H^1} \leq 1$ on a compact oriented surface.

2. Let $\int u = 0$ and $u \in H^1$. Let e_n be the eigenfunctions of the Laplacian with eigenvalues λ_n. Then $u = \sum_n u_n e_n$ in L^2. Prove that

   (a) $u_0 = 0$.

   (b) The eigenvalues of Δ are non-positive. (Denote the largest non-zero one by λ_1.)

   (c) $\|\nabla u\|_{L^2}^2 = -\sum_n |u_n|^2 \lambda_n$

   (d) Deduce the Poincaré inequality.

3. Let f_1, f_2 be smooth positive functions and $p \geq 1$ be any real number. Solve

$$\Delta u = f_1 |u|^p - f_2$$

   on a compact oriented surface using your favourite method.