Lecture 15 - UM 102 (Spring 2021)

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Recap

• Every complex $n \times n$ matrix

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Hermitian linear maps

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