# Lecture 15 - UM 102 (Spring 2021) 

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IISc

## Recap

- Every complex $n \times n$ matrix


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- Three questions:
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