Erratum - Notes on the Wronskian

The Wronskian of two differentiable functions $u_1, u_2 : (a,b) \to \mathbb{R}$ is defined as $W(x) = u_1 u_2' - u_1' u_2$.

If u_1, u_2 are linearly dependent, then $W(x) = 0$ for all x (as can be easily checked), i.e., if $W(x)$ is not identically zero, then u_1, u_2 are linearly independent. So to prove linear *independence*, the Wronskian is a useful tool.

Unfortunately, I was wrong in the class when I asserted the converse. It is *not* true that just because $W(x)$ is identically zero, u_1, u_2 are linearly dependent, i.e., if u_1, u_2 are linearly independent, $W(x)$ can still be identically zero. The counterexample due to Peano is as follows: $x^2, x|x|$ are linearly independent and yet their Wronskian vanishes identically.

However, if u_1, u_2 form a basis of solutions of

$$y'' + Py' + Qy = 0$$

on $\mathbb{R}$ (anyway these exist on all of $\mathbb{R}$), then

Lemma 0.1. *$W(x)$ is nowhere 0 in this case.*

Proof. We calculate as follows.

$$\begin{aligned} W' = u_1 u_2'' - u_1'' u_2 &= u_1(-Pu_2' - Qu_2) - (-Pu_1' - Qu_1)u_2 \\ &= -P(u_1 u_2' - u_2 u_1') = -PW. \end{aligned} \tag{1}$$

Thus $W(x) = W(x_0)e^{-P(x-x_0)}$. Therefore, if $W(x_0) = 0$ for a single point x_0, $W(x) = 0$ identically. Moreover, neither u_1 nor u_2 can be identically 0 (because they have been assumed to form a basis of solutions). Suppose x_0 is a point where $u_1(x_0) \neq 0$. Define $c = -\frac{u_2(x_0)}{u_1(x_0)}$. We claim that if $W(x_1) = 0$ for any x_1 (and thus $W(x) = 0$ identically), then $y(x) := u_2(x) + cu_1(x) = 0$ identically.

Indeed, $y(x_0) = u_2(x_0) + cu_1(x_0) = 0$ by definition of c. Also, $W(x_0) = 0$ (because $W(x_1) = 0$ implies that $W(x)$ vanishes everywhere) implies that $y'(x_0) = u_2'(x_0) - \frac{u_2(x_0)}{u_1(x_0)} u_1'(x_0) = 0$. Moreover, y satisfies the homogeneous equation. Thus $y(x) = 0$ identically by uniqueness of solutions with given initial conditions. (How does one prove this particular statement of uniqueness again? One recasts this second-order ODE as a first order system, brings the matrix to an upper triangular form, etc.) □