HW 1(to be tested on March 18)

1. Let V be a real vector space and $S \subseteq V$ be a subset. Suppose $S \subseteq T \subseteq V$ where T is a subspace. Prove that $L(S) \subseteq T$ where $L(S)$ is the span of V.

2. Let $a < b$ be real numbers. Suppose $w : [a, b] \to \mathbb{R}$ is a continuous function such that $w(t) > 0$ for all $t \in [a, b]$. Verify that $\langle f, g \rangle = \int_a^b w(t) f(t) \bar{g}(t) dt$ is an inner product on the vector space of continuous complex-valued functions on $[a, b]$.

3. Let H be a symmetric positive-definite real $n \times n$ matrix. Let V be a finite-dimensional real vector space with an ordered basis $e_1, \ldots, e_n$. Prove that $\langle v, w \rangle = v^T H w$ is an inner product.

4. Let V be the vector space of complex polynomials of degree ≤ 3. Let $T : V \to V$ be the map $T(p) = p(x + 1)$. Let $D : V \to V$ be the differentiation map. Consider the two ordered bases (the first one for the domain and the second one for the target) : $1, x, x^2, x^3$ and $1, x + 1, x^2 + 2x, x^3$.

   (a) Prove that $1, x + 1, x^2 + 2x, x^3$ is indeed a basis.

   (b) Prove that T is a linear map.

   (c) What is the matrix representing $TD - D^2$ in these bases ?

   (d) Find a basis for the null space of $TD - D^2$. What is the dimension of the null space of $TD - D^2$?

   (e) What about the dimension of its range ?

5. Let V, W be vector spaces over the same field. Let $T : V \to W$ be a linear map. If V is infinite-dimensional, then prove that at least one of $N(T)$ or $R(T)$ is infinite-dimensional.