HW 3 (to be tested on April 1)

1. Prove that there for any positive integer $n \geq 3$, there is no triple of positive integers a, b, c such that $a^n + b^n = c^n$. (Hint: You might find the answer in a margin.)

2. Prove that row-reduction does not change the row space. (A part of this problem is to make this statement rigorous.)

3. Let A be an $m \times n$ matrix. Prove the following properties.

   (a) $(A^T)^T = A$.

   (b) If $c \in \mathbb{F}$, $(cA)^T = cA^T$.

   (c) $(A + B)^T = A^T + B^T$ if B is an $m \times n$ matrix.

   (d) If C is an $n \times p$ matrix, then $(AC)^T = C^T A^T$.

   (e) If A is an invertible $n \times n$ matrix, then $(A^{-1})^T = (A^T)^{-1}$.

4. Prove that the dimension of the row space equals the dimension of the column space of an $m \times n$ matrix. (Hint: Consider the matrix as a linear map, use the idea of free variables, as well as the nullity-rank theorem.)

5. Prove that if F is a multilinear function, then $F(\ldots, v_k + c_1 w_1 + \ldots + c_m w_m, \ldots) = F(\ldots, v_k, \ldots) + c_1 F(\ldots, w_1, \ldots) + c_2 F(\ldots, w_2, \ldots) + \ldots$.

6. Prove that the system $x + y + 2z = 2, 2x - y + 3z = 2, 5x - y + az = 6$ has a unique solution if $a \neq 8$. Find all solutions when $a = 8$.

7. For each of the following, give a proof or a counterexample.

   - If A and B are $n \times n$ invertible matrices, then AB is invertible.
   - If A and B are $n \times n$ invertible matrices, then $A + B$ is invertible.
   - If the product of k square matrices A_i, i.e., $A_1 A_2 \ldots A_k$ is invertible, then each of them is invertible (without using determinants).