HW 5 (to be tested on April 15)

1. If A, B are square matrices, then prove that $tr(A+B) = tr(A) + tr(B)$, $tr(cA) = ctr(A)$, and $tr(AB) = tr(BA)$. Also prove that similar matrices have the same trace. Also prove that there are no $n \times n$ complex matrices X and P such that $XP - PX = \sqrt{-1}\hbar I$.

2. If A is a square matrix, then prove that $p_A(\lambda) = \det(\lambda I - A)$ is a polynomial of degree n such that the coefficient of λ^{n-1} is $-tr(A)$. Prove that $\sum_i \lambda_i = tr(A)$ and that $\det(A) = \lambda_1 \lambda_2 \ldots$.

3. If A, B are invertible square matrices, prove that AB and BA have the same characteristic polynomial.

4. If $T : V \to V$ has the property that T^2 has a nonnegative eigenvalue λ^2 prove that either λ or $-\lambda$ is an eigenvalue of T.

5. Let $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Find a formula for M^n.

6. Determine whether $A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & 3 \end{bmatrix}$ is diagonalisable or not. Find the eigenvalues and eigenspaces. If it is diagonalisable, find a matrix P such that $P^{-1}AP$ is diagonal.

7. Let V be the inner product space of degree ≤ 3 real polynomials with the inner product $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt$. Determine which of the following functions $T : V \to V$ is a symmetric or skew-symmetric linear map.

   (a) $T(f) = f(x)f(-x)$.

   (b) $T(f) = f(-x)$

   (c) $T(f) = f(x) + f(-x)$

   (d) $T(f) = f(x) - f(-x)$