HW 6 (to be tested on April 22)

1. Let $w(t) : (a,b) \to \mathbb{C}$ be a function such that $w(t) = u(t) + \sqrt{-1}v(t)$ where $u, v : (a,b) \to \mathbb{R}$ are differentiable. We then say that $w(t)$ is differentiable and $w' = u' + \sqrt{-1}v'$. Prove that if $w_1(t), w_2(t)$ are two such differentiable complex-valued functions, then so is $w_1(t)w_2(t)$ and its derivative is $w_1'w_2 + w_1w_2'$. Prove that if $w_2 \neq 0$, then $\frac{w_1}{w_2}$ is differentiable and its derivative is $\frac{w_1'w_2 - w_2'w_1}{w_2^2}$. If $g : (a,b) \to (a,b)$ is differentiable, then prove that $w(g(t))$ is also differentiable and its derivative is $w'(g(t))g'(t)$.

2. Prove that $w(t) = w_0e^{\alpha t}(\cos(\beta t) + \sqrt{-1}\sin(\beta)t)$ is the solution of $w' = (\alpha + \sqrt{-1}\beta)w$ and $w(0) = w_0$.

3. Prove that the roots of $D^2 + PD + Q = 0$ are eigenvalues of $A = \begin{bmatrix} -P & -Q \\ 1 & 0 \end{bmatrix}$.

4. Let $P \in \mathbb{R}$ be a constant. Let $A = \begin{bmatrix} -P & \frac{-P^2}{4} \\ 1 & 0 \end{bmatrix}$.

   (a) Prove that $\lambda = -\frac{P}{2}$ is the only eigenvalue and its eigenspace is one-dimensional.

   (b) Let u be an eigenvector of $-\frac{P}{2}$ and let v be one of e_1, e_2 such that u, v form a basis. Prove that in this basis, A is upper-triangular.

   (c) Now solve $\frac{d\vec{y}}{dt} = A\vec{y}$ on $\mathbb{R}$ and prove that the solution space is a two-dimensional real vector space spanned by $e^{\lambda t}$ and $te^{\lambda t}$.

5. Solve $y'' - 2y' + 2y = 0$ with $y(0) = 1$ and $y(\frac{\pi}{2}) = e^{\frac{\pi}{2}}$.