HW 7 (to be tested on May 13)

1. Let $\lambda(\vec{x}) : S \subset \mathbb{R}^n \to \mathbb{R}$, $\vec{f} : S \subset \mathbb{R}^n \to \mathbb{R}^m$ be functions. If $\lim_{\vec{x}\to\vec{a}} \lambda(x) = a$ and $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{b}$ exist, then prove using the $\delta - \epsilon$ definition that $\lim_{\vec{x}\to\vec{a}} \lambda(\vec{x})\vec{f}(\vec{x})$ exists and equals $a\vec{b}$.

2. In each of the following cases, prove (or disprove):

   (a) If A_i are infinitely many open sets, then their intersection $\cap_i A_i$ is open.

   (b) If A_i are infinitely many closed sets, then their intersection $\cap_i A_i$ is closed.

   (c) The set $\{(x, y, z) \in \mathbb{R}^3 \mid |x + y| < 1, |z| < 1\}$ is open.

3. Prove that a set S is closed if and only if $S = Int(S) \cup \partial S$.

4. If $\lim_{(x,y)\to(a,b)} f(x, y) = L$ exists, and $\lim_{x\to a} f(x, y), \lim_{y\to b} f(x, y)$ exist, then prove that $\lim_{x\to a} \lim_{y\to b} f(x, y) = \lim_{y\to b} \lim_{x\to a} f(x, y) = L$.

5. Let $f(x, y) = x \sin(1/y)$ if $y \neq 0$ and $f(x, 0) = 0$. Show that $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ but $\lim_{x\to 0} \lim_{y\to 0} f(x, y) \neq \lim_{y\to 0} \lim_{x\to 0} f(x, y)$. Why does this not contradict the earlier result?

6. Find the limit $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ or prove that it does not exist.

7. (a) Prove that there is no scalar field f such that $\nabla_{\vec{v}} f(\vec{a}) > 0$ for a fixed vector $\vec{a}$ and every non-zero vector $\vec{v}$.

   (b) Give an example of a scalar field f such that $\nabla_{\vec{v}} f(\vec{a}) > 0$ for a fixed vector $\vec{v}$ and every vector $\vec{a}$.