HW 8 (to be tested on May 20)

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Prove that $\lim_{\vec{r}\to\vec{a}} f(\vec{r})$ exists and equals L if and only if for any sequence $\vec{r}_n \to \vec{a}$ (meaning that the components converge individually to the respective components of $\vec{a}$), the sequence $f(\vec{r}_n)$ converges to L.

2. Let $f(x, y) = 3x^2 + y^2$. Why is f differentiable on $\mathbb{R}^2$? Find the points (x, y) and the directions for which the directional derivative of $f(x, y)$ has the largest value if the points (x, y) are restricted to lie on the circle $x^2 + y^2 = 1$.

3. Prove that $f(x, y) = \tan\left(x^2 + y^2 + \frac{\pi}{2}\right)$ when $0 < x^2 + y^2 < 1$ and $f(0, 0) = 0$ is differentiable everywhere on $x^2 + y^2 < 1$ except at $(0, 0)$ (where you need to prove that it is *not* differentiable). Calculate its total derivative linear map at the point $(\frac{1}{2}, \frac{1}{2})$.

4. A differentiable *path* is a vector-valued function $\vec{r}(t) : (a, b) \subset \mathbb{R} \to \mathbb{R}^n$. A regular path is one whose velocity $\vec{r}'(t) \neq 0$ for all $t \in (a, b)$. A curve is the range of a path, i.e., a curve is a subset of $\mathbb{R}^n$. Suppose $\vec{u}(\tau) : (c, d) \to \mathbb{R}^n$ is another path such that it is a *reparametrisation* of $\vec{r}(t)$, i.e., $\tau(t) : (a, b) \to (c, d)$ is 1-1 onto continuously differentiable function such that its inverse is also continuously differentiable, $\tau'(t) > 0$ everywhere, and $\vec{u}(\tau(t)) = \vec{r}(t)$. Firstly, prove that the ranges of the paths, i.e., the curves are the same curve C. Prove that the directional derivative along C, i.e., $\frac{df}{ds}$ is the same at any point (x_0, y_0) on C. In other words, $\frac{df}{ds}$ is a property of the *curve* (the geometric object) and not of any particle moving on it.