Lecture 39 - UM 102 (Spring 2021)

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- Its integral over $x^2 + y^2 \leq 4$ is (in polar coordinates) $\int_0^2 \int_0^{2\pi} (\frac{3}{2}r^6 \sin^(2\theta) \cos(2\theta) + 27r\sin(\theta) - 6r^3\sin(\theta))d\theta rdr = 0$.

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- If a smooth $\vec{F}$ is conservative, i.e., $\vec{F} = \nabla f$ for some smooth f, then $F_i = f_{x_i}$. Thus, $F_{i,x_j} = f_{x_j x_i} = f_{x_i x_j} = F_{j,x_i}$. In other words, in $\mathbb{R}^3$, $\nabla \times \vec{F} = \vec{0}$. So the vanishing of the curl is a *necessary* condition for the vector field to be conservative.

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