Lecture 36 - UM 102 (Spring 2021)

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- Examples: Take $\int_0^2 u^2 du$ with $u = 2x$ to get $\int_0^1 8x^2 dx$. But with $u = 2 - 2x$, $\int_0^2 u^2 du = -\int_1^0 8(1-x)^2 dx = 2\int_0^1 4(1-x)^2 dx$. If we want to use only $[0, 2]$ and $[0, 1]$ instead of the upper and lower limits, then in the second example, $\int_{[0,2]} u^2 du = \int_{[0,1]} 4(1-x)^2 |\frac{du}{dx}| dx$.
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- We want a generalisation for multiple integrals. (Indeed, if we have cylindrical or spherical symmetry, it makes sense to choose other coordinates systems like polar coordinates.)

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- Thus $V_n(1) = V_{n-2}(1) \int \int_D (1 - x^2 - y^2)^{(n-2)/2} dxdy = 2\pi V_{n-2}(1) \int_0^1 (1 - r^2)^{(n-2)/2} rdr = V_{n-2}(1) \frac{2\pi}{n}$.

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