Lecture 38 - UM 102 (Spring 2021)

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- Area (scalar and vector) of regular parametrised surfaces.

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- Area (scalar and vector) of regular parametrised surfaces.
- Scalar line integral and scalar surface integral.

Reparametrisation invariance

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- Theorem: The surface integral is reparametrisation invariant.
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Flux/Vector surface integral

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- Rigorously, if $S \subset \mathbb{R}^3$ is a regular parametrised surface, and $\vec{F}$ is a bounded vector field on S, then the flux of $\vec{F}$ through S is defined to be

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- Rigorously, if $S \subset \mathbb{R}^3$ is a regular parametrised surface, and $\vec{F}$ is a bounded vector field on S, then the flux of $\vec{F}$ through S is defined to be $\int\int_T \vec{F}.(\vec{r_u} \times \vec{r_v})dudv$.

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