Lecture 2 - UM 102 (Spring 2021)

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- $T(\sum_i c_i v_i) = \sum_i T(c_i v_i)$: We prove by induction. For $n = 1$, it follows from definition. Assume truth for n. For $n + 1$, $T(\sum_{i=1}^{n+1} c_i v_i) = T(\sum_{i=1}^{n} c_i v_i + c_{n+1} v_{n+1}) = \sum_{i=1}^{n} c_i T(v_i) + c_{n+1} T(v_{n+1})$.

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- Different choices of ordered bases give rise to *different* matrices representing the *same* linear map.

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- Thus the matrix is $[T] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

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- If we choose ordered bases $\{e_i\}, \{f_j\}, \{g_k\}$ for f.d vector spaces V, W, X, and if $T : V \to W$, $U : W \to X$ are linear maps, then then we get two matrices $[T]$ and $[U]$ representing the maps.
- It turns out that $U \circ T$ is represented by $[U][T]$ where multiplication is in the sense of *matrix multiplication*, i.e., $([A][B])_{ij} = \sum_k [A]_{ik}[B]_{kj}$: Indeed, $U(T(e_i)) = U(\sum_j T_{ji} f_j) = \sum_j T_{ji} U(f_j) = \sum_j T_{ji} \sum_k U_{kj} g_k = \sum_{j,k} U_{kj} T_{ji} g_k$.

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- $(A+B)C = AC + BC$ and $C(A+B) = CA + CB$ whenever it makes sense.
- The simplest proof is to interpret each of the matrices as linear maps between appropriate vector spaces and use the fact that $[U \circ T] = [U][T]$.

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