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Lecture 3 - UM 102 (Spring 2021)

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- This proof is like taking a dot product with a bunch of vectors and isolating each component.
- So it is fruitful to define the notion of a dot product on arbitrary vector spaces (over $\mathbb{R}$ or $\mathbb{C}$. This notion does not make sense for all fields).

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- $f(t) = \|v\|^2 + t^2\|w\|^2 + 2t\langle v, w\rangle$. $f'(t) = 0$ implies that $t = -\frac{\langle v,w\rangle}{\|w\|^2}$.

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- Hence $\langle v, w\rangle^2 \leq \|v\|^2\|w\|^2$. Equality holds precisely when $v + tw = 0$, i.e., $v = -tw$.
- For the complex case, choose $t = -\frac{\langle v,w\rangle}{\|w\|^2}$ as before.