Lecture 4 - UM 102 (Spring 2021)

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- Wrote the expression for an inner product (for a real vector space) in terms of a basis using positive-definite matrices.
- Stated and proved the Cauchy-Schwarz inequality.
- Defined norms and proved their properties (including the triangle inequality).

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- An important result is: In an inner product space $(V, \langle , \rangle)$, an orthogonal set of nonzero elements is linearly independent. In particular, if V is f.d with $dim(V) = n$, any orthogonal set of nonzero elements of size n forms a basis.

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  - The set $\{1, x, x^2\}$ is not orthogonal under the integration inner product.

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- It turns out that in a certain function space (larger than continuous functions), e^{ikx} form an orthonormal "basis" of sorts. The analogue of the theorem above was discovered by Fourier and Parseval. It forms the basis for Fourier's technique of solving certain differential equations.

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- Thus, every finite-dimensional inner product space has an orthogonal basis.
- By dividing each element by its norm, we can convert an orthogonal basis to an orthonormal basis.

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- More generally, it turns out that $y_n = \frac{n!}{(2n)!} \frac{d^n (t^2-1)^n}{dt^n}$. The Legendre polynomials are $P_n(t) = \frac{(2n)!}{2^n (n!)^2} y_n(t)$.