Lecture 6 - UM 102 (Spring 2021)

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- If $b = 0$, then the system $AX = 0$ is called a *homogeneous* system.
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