Lecture 9 - UM 102 (Spring 2021)

Vamsi Pritham Pingali

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- The (signed) area is $\vec{v} \times \vec{w} = (ad - bc)\hat{k}$.

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  - Normalisation: $F(e_1, \ldots, e_n) = 1$.

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- We can prove by induction on n that *any* permutation can be obtained by a finite number of *interchanges*. Indeed, it is trivial for $n = 1$. One of j_i corresponds to n. Suppose it is j_k. Now $[1, 2, \ldots, n-1] \to [j_1, \ldots, j_{k-1}, j_n, j_{k+1}, \ldots, j_{n-1}]$ is a permutation of $n-1$ things. By the induction hypothesis, it can be obtained using a finite number of interchanges. That is, $[1, 2 \ldots, n] \to [j_1, \ldots, j_{k-1}, j_n, j_{k+1}, \ldots, j_{n-1}, j_k = n]$ can be obtained that way. Now interchange j_k with j_n to get the desired permutation.
- Using the above result we see that $d(e_{j_1}, \ldots, e_{j_n}) = (-1)^K d(e_1, \ldots, e_n) = (-1)^K$ and

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- Using the above result we see that $d(e_{j_1}, \ldots, e_{j_n}) = (-1)^K d(e_1, \ldots, e_n) = (-1)^K$ and $f(e_{j_1}, \ldots, e_{j_n}) = (-1)^K f(e_1, \ldots, e_n) = d(e_{j_1}, \ldots, e_{j_n}) f(e_1, \ldots, e_n)$.

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- Using the above result we see that $d(e_{j_1}, \ldots, e_{j_n}) = (-1)^K d(e_1, \ldots, e_n) = (-1)^K$ and $f(e_{j_1}, \ldots, e_{j_n}) = (-1)^K f(e_1, \ldots, e_n) = d(e_{j_1}, \ldots, e_{j_n}) f(e_1, \ldots, e_n)$.
- Thus $f(v_1, \ldots, v_n) = \sum c_{1j_1} \ldots d(e_{j_1}, \ldots, e_{j_n}) f(e_1, \ldots, e_n) = d(v_1, \ldots, v_n) f(e_1, \ldots, e_n)$.