Lecture 10 - UM 102 (Spring 2021)

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Recap

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- Defined them as multilinear alternating normalised maps taking tuples of vectors to scalars.
- Proved uniqueness of the determinant function.

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- which can be column-transformed to $= ad\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$.
- which equals $ad - bc$.

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- By scaling, $\det(U) = u_{11}\det(U')$ where U' has e_1 in the first column. By multilinearity, i.e., column transformations, we can "clear" the first row.

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- Thus $\det(U) = u_{11} \det(U'')$ where U'' is the $(n-1) \times (n-1)$ matrix obtained by deleting the first row and first column. It is upper triangular.
- We are done by the induction hypothesis.

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- Scaling: If $v_i \to tv_i$, for $j \neq i$ each of the A_{1j} scales by t by the induction hypothesis. For $j = i$ A_{1i} remains unchanged by c_{1i} scales by t. Thus every term in the definition scales by t.
- Linearity: If $v_i \to v_i + w$, for $j \neq i$ as before, A_{1j} is linear. For $j = i$, as before, the coefficient is linear. We are done.
- Normalisation: $\det(e_1, \ldots, e_n) = A_{11} = 1$ by the induction hypothesis.
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