Lecture 24 - UM 102 (Spring 2021)

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IISc

Recap

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- The reason differentiable 1-variable functions

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- The reason differentiable 1-variable functions are continuous

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- Take $\vec{h} = h\vec{v}$ for a small enough h. Then $\frac{f(\vec{a}+h\vec{v})-f(\vec{a})}{h} - Df_{\vec{a}}(\vec{v}) = \frac{\||h|\|\vec{v}\|\| E(\vec{a},h\vec{v})}{h}$.

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- This means that $\nabla_{\vec{v}}f$ exists and equals $Df(\vec{v})$.
- Since Df is linear, $Df(\vec{v}) = \sum_i v_i Df(\vec{e_i}) = \sum_i v_i \nabla_{\vec{e_i}} f = \langle \nabla f, \vec{v} \rangle$.

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