# Lecture 26 - UM 102 (Spring 2021)

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- Defined differentiability as the linear approximation.
- Proved that differentiability implies continuity.
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- More rigorously, $f(a+h,b+k) - f(a,b) - f_x(a,b)h - f_y(a,b)k = (I - f_x(a,b)h) + (II - f_y(a,b)k)$. Hence, when $\|(h,k)\| < \delta$ (which immediately implies that $|h| < \delta, |k| < \delta$), then by continuity of $f_x, f_y$, $|f_x(a+\theta_1, b+k) - f_x(a,b)| < \frac{\epsilon}{2}$ and $|f_y(a, b+\theta_2) - f_y(a,b)| < \frac{\epsilon}{2}$. Thus $|(I - f_x(a,b)h)| < |h|\frac{\epsilon}{2}$ and $|II - f_y(a,b)k| < |k|\frac{\epsilon}{2}$.
- Thus, $\frac{|f(a+h,b+k) - f(a,b) - f_x(a,b)h - f_y(a,b)k|}{\|(h,k)\|} < \epsilon$. This implies the result in this case.

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- For each of the partials, we can replace them by their values at $\vec{a}$ at the cost of an error $\frac{\epsilon}{n}$ provided $\vec{h}$ is small enough. The same manipulations as before show what we need.