Lecture 27 - UM 102 (Spring 2021)

Vamsi Pritham Pingali

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Recap

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- Proved a theorem

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- Proved a theorem about limits

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- Proved a theorem about limits that implied that it is enough to “test” a limit

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- Suppose $t_0 \in (a,b)$ is a point where $x_1(t), x_2(t), \ldots$ are differentiable functions and f is differentiable at $\vec{r}(t_0)$.
- Then $h(t)$ is differentiable at t_0 and $h'(t_0) = \langle \nabla f(\vec{r}(t_0)), \vec{r}'(t_0) \rangle = \nabla_{\vec{r}'(t_0)} f(\vec{r}(t))$.

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