Lecture 29 - UM 102 (Spring 2021)

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- Defined paths, regular paths, and curves. Defined the notion of directional derivative along a regular path $\frac{df}{ds}$ as $\frac{1}{\|\vec{r}'(t_0)\|} \langle \nabla f(\vec{r}(t_0)), \vec{r}'(t_0) \rangle$.

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- So $\frac{g(t_0+h)-g(t_0)}{h} = \frac{\langle \nabla f(\vec{r}(t_0)), \vec{r}(t_0+h)-\vec{r}(t_0)\rangle}{h} + \frac{\|\vec{r}(t_0+h)-\vec{r}(t_0)\|}{h} E(\vec{a}, \vec{r}(t_0 + h) - \vec{r}(t_0))$.

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- So $\frac{g(t_0+h)-g(t_0)}{h} = \frac{\langle \nabla f(\vec{r}(t_0)), \vec{r}(t_0+h) - \vec{r}(t_0) \rangle}{h} + \frac{\|\vec{r}(t_0+h) - \vec{r}(t_0)\|}{h} E(\vec{a}, \vec{r}(t_0+h) - \vec{r}(t_0))$. This goes to the correct answer as $h \to 0$. Indeed $\left|\frac{\|\vec{r}(t_0+h) - \vec{r}(t_0)\|}{h}\right| E \to 0$.

Level sets and tangent planes

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- Returning back to this concept in greater generality, whenever $f : \mathbb{R}^n \to \mathbb{R}$ is a function, the set $\vec{r} \in \mathbb{R}^n$ such that $f(\vec{r}) = c$ is called a *level set* of f. (

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- If f is C^1, and on the *entire* level set $f^{-1}(c)$, $\nabla f \neq \vec{0}$ (a *regular* level set), then it turns out (by a theorem called the implicit function theorem) that near any point on this level set the level set can be treated as a graph of a function. In particular, the tangent planes exist at any point.

Level sets and tangent planes

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Vector fields

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- Proof: $\|\vec{F}(\vec{a}+\vec{h}) - \vec{F}(\vec{a}) - D\vec{F}_{\vec{a}}(\vec{h})\| < \|\vec{h}\|\frac{\epsilon}{2}$ whenever $0 < \|\vec{h}\| < \delta < 1$. By the triangle inequality $\|\vec{F}(\vec{a}+\vec{h}) - \vec{F}(\vec{a})\| < \|D\vec{F}_{\vec{a}}(\vec{h})\| + \frac{\epsilon}{2}$.
- At this juncture, we prove a useful linear algebraic lemma: Suppose A is an $m \times n$ matrix and $\vec{v} \in \mathbb{R}^n$. Then $\|A\vec{v}\| \leq C_A\|\vec{v}\|$ where $C_A = \sum_i \|A_i\|$ (where A_i is the i^{th} row of A).
- Proof of lemma: $\|A\vec{v}\| = \|\sum_i \langle A_i, \vec{v}\rangle\| \leq \sum_i |\langle A_i, \vec{v}\rangle e_i| \leq \sum_i \|A_i\|\|\vec{v}\|$ by Cauchy-Schwarz.

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- Returning back to the proof of the theorem, $\|\vec{F}(\vec{a}+\vec{h})-\vec{F}(\vec{a})\| < \|\vec{h}\| C_{D\vec{F}_{\vec{a}}} + \frac{\epsilon}{2} < \epsilon$ if $\|\vec{h}\|$ is small enough. $\square$