Lecture 30 - UM 102 (Spring 2021)

Vamsi Pritham Pingali

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Recap

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- Proved the chain rule

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- Proved the chain rule for scalar fields.

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- Did level sets and

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- Directional derivatives and differentiability for vector fields.

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- Proved the chain rule for scalar fields.
- Did level sets and tangent planes of regular level sets.
- Directional derivatives and differentiability for vector fields. Derivative matrix.

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- Proved the chain rule for scalar fields.
- Did level sets and tangent planes of regular level sets.
- Directional derivatives and differentiability for vector fields. Derivative matrix. Differentiability implies continuity.

Chain rule for scalar fields

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