Lecture 31 - UM 102 (Spring 2021)

Vamsi Pritham Pingali

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- Stated Clairut's theorem on mixed partials.
- Proved the second order Taylor theorem and used it to prove the second derivative test for local extrema in one-variable calculus.

An example

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Second-derivative test

- Before formulating

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- Before formulating a second-derivative test for local extrema,

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- Let $\vec{a}$ be a critical point of f. Suppose f is C^3 in a neighbourhood of $\vec{a}$ (that is, the first, second, and third partials exist in a neighbourhood of a and are continuous there; by Clairut, the mixed partials are equal).
- Theorem: Under the above assumptions, for all $\vec{h}$ lying in a certain neighbourhood of $\vec{0}$, $|f(\vec{a}+\vec{h}) - f(\vec{a}) - \nabla_{\vec{h}} f(\vec{a}) - \frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) h_i h_j| \leq C\|h\|^3$ for some $C > 0$.
- Proof: Consider $u(t) = f(\vec{a}+t\vec{h})$. By any application of the chain rule and properties of continuity, we see that $u(t)$ is C^3 in $(-\epsilon, \epsilon)$ for some $\epsilon > 0$. Applying a precise version of the one-variable Taylor theorem, it turns out that $|u(t) - u(0) - u'(0)t - \frac{u''(0)}{2}t^2| \leq C|t^3|$ in a neighbourhood of $t = 0$ and C does not depend on $\vec{h}$. Now $u'(0) = \nabla_{\vec{h}} f(\vec{a})$. In fact, $u'(t) = \sum_i \frac{\partial f}{\partial x_i}(\vec{a}+t\vec{h})h_i$. Thus $u''(0) = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) h_i h_j$.

Second-order Taylor expansion

- Let $\vec{a}$ be a critical point of f. Suppose f is C^3 in a neighbourhood of $\vec{a}$ (that is, the first, second, and third partials exist in a neighbourhood of a and are continuous there; by Clairut, the mixed partials are equal).
- Theorem: Under the above assumptions, for all $\vec{h}$ lying in a certain neighbourhood of $\vec{0}$, $|f(\vec{a}+\vec{h}) - f(\vec{a}) - \nabla_{\vec{h}} f(\vec{a}) - \frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) h_i h_j| \leq C\|h\|^3$ for some $C > 0$.
- Proof: Consider $u(t) = f(\vec{a} + t\vec{h})$. By any application of the chain rule and properties of continuity, we see that $u(t)$ is C^3 in $(-\epsilon, \epsilon)$ for some $\epsilon > 0$. Applying a precise version of the one-variable Taylor theorem, it turns out that $|u(t) - u(0) - u'(0)t - \frac{u''(0)}{2}t^2| \leq C|t^3|$ in a neighbourhood of $t = 0$ and C does not depend on $\vec{h}$. Now $u'(0) = \nabla_{\vec{h}} f(\vec{a})$. In fact, $u'(t) = \sum_i \frac{\partial f}{\partial x_i}(\vec{a} + t\vec{h}) h_i$. Thus $u''(0) = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) h_i h_j$. Now replace t with $|h|$ and h with $\frac{h}{|h|}$ to get the result.