Lecture 32 - UM 102 (Spring 2021)

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Recap

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- Defined saddle points.

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- Defined local and global extrema in more than one variable.
- Defined critical points and proved the first derivative test.
- Did an example of global extrema.
- Defined saddle points.
- Proved the second-order Taylor theorem.

Second-derivative test

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- Answer: Since H is Hermitian, it is diagonalisable as $H = U^{\dagger} D U$ for some unitary U. Thus $h^T H \bar{h} = h^T U^{\dagger} D U \bar{h} = \sum_i \lambda_i |(U\bar{h})_i|^2 = \sum_i |y_i|^2 \lambda_i$ where $y = U\bar{h}$. Thus this expression is positive for all h if and only if it is so for all y if and only $\lambda_i > 0$ for all i. Likewise, H is negative-definite if and only if all the eigenvalues are negative. It is invertible

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- Let $u(t) : [a, b] \to [c, d]$ be a C^1 function such that $u'(t) \neq 0$ for all $t \in [a, b]$. u is $1-1$ because either $u'(t) > 0$ for all t or $u'(t) < 0$ for all t. So t is a function of u and it turns out that t is C^1 in u. Such a u is called a change of parameter. If $u' > 0$ for all t,

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Reparametrisation

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Back to Work

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