Lecture 34 - UM 102 (Spring 2021)

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Recap

Recap

- Second-derivative test

Recap

- Second-derivative test and an example.

Recap

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- Line integrals,

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- Second-derivative test and an example.
- Line integrals, their properties, and reparametrisation invariance.

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- Proof:

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