Lecture 22 - UM 102 (Spring 2021)

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Recap

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Limit and Continuity laws

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- If $\vec{f}, \vec{g}$ are scalar-valued, $\vec{g}(\vec{x})$ is not zero in a neighbourhood of $\vec{a}$ (intersected with S), and $c \neq 0$, then $\lim_{\vec{x}\to\vec{a}} \frac{f}{g} = \frac{b}{c}$.

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- If $\vec{f}, \vec{g}$ are scalar-valued, $\vec{g}(\vec{x})$ is not zero in a neighbourhood of $\vec{a}$ (intersected with S), and $c \neq 0$, then $\lim_{\vec{x}\to\vec{a}} \frac{f}{g} = \frac{b}{c}$.
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- If $\vec{f}, \vec{g}$ are scalar-valued, $\vec{g}(\vec{x})$ is not zero in a neighbourhood of $\vec{a}$ (intersected with S), and $c \neq 0$, then $\lim_{\vec{x}\to\vec{a}} \frac{f}{g} = \frac{b}{c}$.
- The same laws hold for continuity too.

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- Without loss of generality

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Proofs of 4,5

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- $|\|\vec{f}(\vec{x})\| - \|\vec{b}\|| \leq \|\vec{f}(\vec{x}) - \vec{b}\|$ by the triangle inequality.

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Examples

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$\vec{f}(\vec{a}+\vec{h}) = \vec{f}(\vec{a}) + \vec{f}(\vec{h}) = \vec{f}(\vec{a}) + \sum_i h_i f(\vec{e_i})$

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- Thus, $\sin(x^2y)$, $\ln \frac{x-y}{x+y}$, $\ln \cos^2(x^2 + y^2)$ etc, are continuous wherever they are defined.

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- Unlike 1-variable calculus, we can ask "How fast does T change when we move a little in a certain direction?" The answer can of course depend on the direction.
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