1 Recap

- Change of variables formula and examples.

2 Parametrisation of surfaces

Let $T \subset \mathbb{R}^2$ be a bounded region whose boundary is a piecewise C^1 Jordan curve (i.e., simple closed regular curve).
Def: A parametrised surface is (the range of) a piecewise C^1 map $\vec{r}(u,v) : T \to \mathbb{R}^3$ that is $1-1$ on the interior.
Example: $\vec{r}(u,v) = (\sin(u)\cos(v), \sin(u)\sin(v), \cos(u))$ where $(u,v) \in T = [0,\pi] \times [0,2\pi]$. The image is the unit sphere. The map is not $1-1$ on the boundary of T. This surface is a "closed" surface, i.e., it has no "boundary". On the other hand, if $T = [0, \frac{\pi}{2}] \times [0, 2\pi]$ then it is a hemisphere whose boundary is a circle.
Def: A closed parametrised surface is a set such that near every point, it can be written as the image of a $1-1$ piecewise C^1 parametrised surface from an open subset of $\mathbb{R}^2$ to $\mathbb{R}^3$. So the sphere is a closed parametrised surface. (On the other hand, a hemisphere is *not* a closed surface.)

Some more examples:

- Example: $\vec{r}(u,v) = (\cos(u), \sin(u), v)$ where $(u,v) \in [0,2\pi] \times [0,1]$. It is a right-circular cylinder with boundary being two circles. (It is not a closed parametrised surface.)

- Example: $\vec{r}(u,v) = (v\sin(\alpha)\cos(u), v\sin(\alpha)\sin(u), v\cos(\alpha))$ where $(u,v) \in [0,2\pi] \times [0,1]$ is a right-circular cone with cone angle α. It is a (non-closed) parametrised surface with boundary as a circle.

Suppose $dudv$ is an infinitesimal area element in the $u-v$ plane. Then the parallelogram formed in $\mathbb{R}^3$ has sides $\vec{r}_u du = du(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u})$ and $\vec{r}_v dv = dv(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v})$. The infinitesimal area is $\|\vec{r}_u \times \vec{r}_v\| dudv$. Just as for regular curves, we define regular surfaces as those with $\vec{r}_u \times \vec{r}_v \neq 0$ everywhere. Such surfaces are also called "smooth". For a cone, $\vec{r}_u = v(-\sin(\alpha)\sin(u), \sin(\alpha)\cos(u), 0)$ and $\vec{r}_v = (\sin(\alpha)\cos(u), \sin(\alpha)\sin(u), \cos(\alpha))$. Thus $\vec{r}_u \times \vec{r}_v = \vec{0}$ when $v = 0$. The vertex of the cone is not a smooth point.
Example: Suppose $z = f(x,y)$ where f is a C^1 function, $\vec{r}(u,v) = (u, v, f(u,v))$. Then $\vec{r}_u = (1, 0, f_u)$ and $\vec{r}_v = (0, 1, f_v)$. Thus $\vec{r}_u \times \vec{r}_v = (-f_u, -f_v, 1)$ and is hence never zero. So the graph of a C^1 function is a regular parameterised surface.
Warning: For instance, if we take $z = \sqrt{1 - x^2 - y^2}$, the function is not differentiable at $x = y = 0$. The problem here lies with this *particular* parametrisation because we already saw the the sphere can be parametrised even near the equator as a regular surface. So the choice of a parametrisation is important.

Proposition: $\vec{r}_u \times \vec{r}_v$ is normal to the surface.
Proof: Indeed, let $(u(t), v(t))$ be a C^1 path on the surface passing through p. Then $\frac{d\vec{r}}{dt} = \vec{r}_u u' + \vec{r}_v v'$. Thus $\vec{r}'$ is perpendicular to $\vec{r}_u \times \vec{r}_v$. □
This is another way to study normals to regular surfaces. The same regular surface can

be given in two ways: $F(x,y,z) = 0$ and as $(x(u,v), y(u,v), z(u,v))$. The normals are ∇F and $\vec{r}_u \times \vec{r}_v$. They point in the same direction for sure. But their magnitudes can be different! (Indeed, if they are the same, then simply choose F to $2F$ and the magnitude doubles!)
For a graph, $F(x,y,z) = z - f(x,y)$ gives the same normal as the previous one. The infinitesimal area vector is $d\vec{A} = \vec{r}_u \times \vec{r}_v dudv$. The area of a parametric surface is $\int\int_T \|\vec{r}_u \times \vec{r}_v\| dudv$. For instance, for a graph $z = f(x,y)$, the area is $\int\int \sqrt{1+f_x^2+f_y^2} dxdy$.

An example and scalar line integrals: Hemisphere: $\vec{r} = (r\sin(u)\cos(v), r\sin(u)\sin(v), r\cos(u))$ and hence $\vec{r}_u = (r\cos(u)\cos(v), r\cos(u)\sin(v), -r\sin(u))$, $\vec{r}_v = (-r\sin(u)\sin(v), r\sin(u)\cos(v), 0)$. So $\|\vec{r}_u \times \vec{r}_v\| = r^2\sin(v)$. Therefore, the area is $2\pi r^2$.
A digression: If $\vec{\gamma}(t)$ is a piecewise C^1 regular path, and $f(x,y,z)$ is a bounded function along the path, then $\int f ds = \int_a^b f(\gamma(t))\|\gamma'(t)\|dt$ is called the scalar line integral of f along γ. (For instance, the charge on a wire.)

A crucial point: The scalar line integral is reparametrisation invariant: If $t(\tau)$ is a reparametrisation, then $\|\frac{d\gamma}{d\tau}\| = \|\gamma'(t)\||\frac{dt}{d\tau}|$. Thus the "new" integral is $\int_{[c,d]} f(\gamma(t(\tau)))\|\gamma'(t)\||\frac{dt}{d\tau}|d\tau$ which by the change of variables formula equals the "old" integral.