1 Recap

- Parametrised surfaces.
- Scalar line integral (its reparametrisation invariance is left as an exercise).

2 Surface integrals

Let $S = \vec{r}(T)$ be a parametrised surface and let f be a bounded scalar field on S. Then $\int\int_S f dA := \int\int_T f(\vec{r}(u,v))\|\vec{r}_u \times \vec{r}_v\| dudv$. When $f = 1$, we get the area. (Akin to the length of a regular curve in the case of line integrals.)
Centre of mass: If f is the density, then $x_{CM} = \int\int x f dA$ and likewise for other coordinates. For instance, for a cone $\vec{r} = (v\sin(\alpha)\cos(u), v\sin(\alpha)\sin(u), v\cos(\alpha))$ where $(u,v) \in [0,2\pi] \times [0,l]$, we see that $dA = v\sin(\alpha)dudv$. Thus if $f = 1$ (uniform density), then $\int\int z dA = \int_0^l \int_0^{2\pi} v^2 \cos(\alpha)\sin(\alpha)dudv = \frac{\pi}{3}\sin(2\alpha)$. It is easy to see that $x_{CM} = y_{CM} = 0$. Thus the centre of mass can lie outside the surface.

Let $\vec{r}(u,v)$ be a piecewise C^1 parametrised surface defined on $T \subset \mathbb{R}^2$. Let $\vec{G}(s,t) = (u(s,t), v(s,t)) : T' \to T$ be a C^1 map that is $1-1$ onto on the interiors. Assume that the Jacobian J of G is nowhere 0 on the interior. Then $\vec{R}(s,t) = \vec{r}(\vec{G}(s,t))$ is called a reparametrisation.
Theorem: $\vec{R}_s \times \vec{R}_t = \vec{r}_u \times \vec{r}_v J$.
Proof: By the chain rule, $\vec{R}_s = \vec{r}_u u_s + \vec{r}_v v_s$, $\vec{R}_t = \vec{r}_u u_t + \vec{r}_v v_t$. Thus $\vec{R}_s \times \vec{R}_t = \vec{r}_u \times \vec{r}_v (u_s v_t - v_s u_t)$, and since $J = u_s v_t - v_s u_t$, we are done. □
Theorem: The surface integral is reparametrisation invariant.
Proof: $\int\int_{\vec{r}(T)} f dA = \int\int_T f\|\vec{r}_u \times \vec{r}_v\| dudv$. By the change of variables formula, this integral equals $\int\int_{T'} f\|\vec{r}_u \times \vec{r}_v\||J| dsdt$. This is precisely the surface integral using the other parametrisation. □

Consider a fluid (can be charged too) moving through space with the velocity vector field $\vec{V}(x,y,z,t)$. If its density is ρ, the amount of fluid per unit area per unit time moving along $\vec{V}$ is $\vec{J} = \rho\vec{V}$ (the flux density or the current vector). The amount per unit time that moves across an infinitesimal surface element $d\vec{A}$ is $\vec{J}.d\vec{A}$. This quantity is the infinitesimal flux. Likewise, if $\vec{E}$ is the electric field, $\vec{E}.d\vec{A}$ is also called flux (roughly measures the "number of lines of force" going through the surface element). Rigorously, if $S \subset \mathbb{R}^3$ is a regular parametrised surface, and $\vec{F}$ is a bounded vector field on S, then the flux of $\vec{F}$ through S is defined to be $\int\int_T \vec{F}.(\vec{r}_u \times \vec{r}_v)dudv$. It is certainly not immediately clear as to whether this quantity is reparametrisation invariant.

As before, if $\vec{G}(s,t), \vec{R}(s,t) = \vec{r}(\vec{G}(s,t))$ are reparametrisation data , then $\int\int_{\vec{r}(T)} \vec{F}.d\vec{A} = \int\int_T \vec{F}.(\vec{r}_u \times \vec{r}_v)dudv = \int\int_{T'} \vec{F}.(\vec{r}_u \times \vec{r}_v)|J| dsdt$. However, $\vec{R}_s \times \vec{R}_t = (\vec{r}_u \times \vec{r}_v)J$. Therefore, there is a sign discrepancy. If $J > 0$ throughout, then $|J| = J$ and the flux integral is reparametrisation invariant. If $J < 0$ throughout, then the flux changes sign. The choice (outward vs inward) of normal is thus important (akin to the vector line integral).
Let S be the unit upper hemisphere parametrised by $(\sin(u)\cos(v), \sin(u)\sin(v), \cos(u))$

where $(u,v) \in T = [0, \frac{\pi}{2}] \times [0, 2\pi]$. Then $\vec{r}_u \times \vec{r}_v = \sin(u)(\sin(u)\cos(v), \sin(u)\sin(v), \cos(u))$. Let $\vec{F} = x\hat{i} + y\hat{j}$. The flux of $\vec{F}$ across S is $\int_0^{2\pi} \int_0^{\pi/2} \sin^3(u) dv du = 0$.

3 Stokes' theorem

We want to generalise Green's theorem to integrals over surfaces.
Theorem: Let S be a C^1 regular parametrised oriented surface $S = \vec{r}(T)$ where $T \subset \mathbb{R}^2$ is an open set in $u-v$ plane bounded by a regular simple closed curve I. Assume that $\vec{r}$ is actually C^2 on an open set containing T. Let C be the curve $\vec{r}(I)$. Let P, Q, R be C^1 scalar fields on S. Let $\vec{F} = (P, Q, R)$ and $\nabla \times \vec{F} = (R_y - Q_z, P_z - R_x, Q_x - P_y)$. Suppose C is oriented in the following manner: The velocity of C, i.e., $\vec{w}$ is such that $(\vec{r}_u \times \vec{r}_v) \times \vec{w}$ points "along" surface, i.e., if you travel a tiny bit along this vector, you are closer to the surface than if you travel in the opposite direction.
Then $\int \int_S (\nabla \times \vec{F}).\vec{dA} = \int_C \vec{F}.\vec{dr}$.
The line integral is sometimes called the *circulation* of $\vec{F}$ because if we consider $\vec{F} = (-y, x, 0)$ and C as the unit circle, then the line integral is non-zero whereas for $\vec{F} = (x, y, 0)$ it is zero. James Clerk Maxwell called $\nabla \times \vec{F}$ as the "curl" of $\vec{F}$ (because it is like the "circulation density").

It is easy to see that if S is a planar surface, then Stokes=Green.
The proof of Stokes: By linearity in $\vec{F}$ and the symmetry of the expression, it is enough to prove it for $\vec{F} = P\hat{i}$. Now $\nabla \times \vec{F} = (0, P_z, -P_y)$ and hence $\nabla \times \vec{F}.\vec{dA} = -P_y(x_u y_v - x_v y_u) + P_z(z_u y_v - z_v y_u)$ which is $(Px_v)_u - (Px_u)_v$ (HW). Apply Green to $\int \int_T ((Px_v)_u - (Px_u)_v) du dv$ to get $\int_C (PX_u du + PX_v dv) = \int_C P dx$. $\square$