# 1 Recap

- Scalar surface integral.
- Vector surface integral/flux through oriented surfaces and Stokes' theorem (Orientation: $(\vec{r}_u \times \vec{r}_v) \times \vec{\gamma}'(t)$ points "along" the surface. An easier way is: Parametrise the boundary of the $u - v$ region $T$ as $u(t), v(t)$ in the anti-clockwise direction. Then the correct parametrisation of the boundary of the surface is $\vec{r}(u(t), v(t))$.)

Example: Let $\vec{F} = (z^2, -3xy, x^3y^3)$ and $S$ be a part of $z = 5-x^2-y^2$ above $z = 1$ with the upwards orientation. Calculate $\int\int_S (\nabla \times \vec{F}).d\vec{A}$. By the way, we did not prove that if the same set is regularly parametrised in two different ways, then they are reparametrisations of each other. This fact is true and requires stuff that is beyond the current scope. Parametrise $S$ as $(x, y, 5 - x^2 - y^2)$ where $x^2 + y^2 \leq 4$. This has the right orientation. Indeed, $\vec{r}_x \times \vec{r}_y = (2x, 2y, 1)$ which points upward. The boundary is a circle $x^2+y^2 = 4$. A correct oriented parametrisation is $(2\cos(t), 2\sin(t), 1)$. Thus by Stokes, the desired integral is $\int_0^{2\pi} (4\sin^2(t), -12\sin(t)\cos(t), 64\sin^3(t)\cos^3(t)).(-2\sin(t), 2\cos(t), 1)dt = 0$. We can verify Stokes by calculating the given thing directly: $\nabla \times \vec{F} = (3x^3y^2, -(3x^2y^3 - 2z), -3y)$. Thus $\nabla \times \vec{F}.d\vec{A} = dxdy(6x^4y^2 - 2y(3x^2y^3 - 2z) - 3y) = dxdy(6x^4y^2 - 6x^2y^4 + 6y(5 - x^2 - y^2) - 3y)$. Its integral over $x^2 + y^2 \leq 4$ is (in polar coordinates) $\int_0^2 \int_0^{2\pi} (\frac{3}{2}r^6 \sin^(2\theta)\cos(2\theta) + 27r\sin(\theta) - 6r^3\sin(\theta))d\theta r dr = 0$.

Interpretation of curl: The curl does not simply measure how much a vector field swirls around. For instance, if $\vec{F} = (-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0)$, then $\nabla \times \vec{F} = \vec{0}$ (away from the origin of course). However, the picture of the vector field tells a different story. Likewise, if $\vec{F} = (x^2 - y^2)(x, y, 0)$, despite pointing radially outward (in the $x - y$ plane) $\nabla \times \vec{F} \neq \vec{0}$. It measures the infinitesimal circulation near a point. For instance, in the first example above, As we go outward, the vector field gets smaller at a very particular rate. In the second example, the vector field is asymmetric in magnitude ( in $x, y$). If a smooth $\vec{F}$ is conservative, i.e., $\vec{F} = \nabla f$ for some smooth $f$, then $F_i = f_{x_i}$. Thus, $F_{i,x_j} = f_{x_j x_i} = f_{x_i x_j} = F_{j,x_i}$. In other words, in $\mathbb{R}^3$, $\nabla \times \vec{F} = \vec{0}$. So the vanishing of the curl is a *necessary* condition for the vector field to be conservative.

Unfortunately, it is not sufficient. For instance, $\vec{F} = (-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2})$ in $\mathbb{R}^2$ has zero curl, but $\int_C \vec{F}.d\vec{r}$ over the unit circle is $2\pi \neq 0$. So the shape of the region is important. In fact, it turns out that it is sufficient on simply connected regions. Akin to $\nabla \times \vec{F}$, one can naively define the "dot product", i.e., $\nabla.\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$. This quantity is called the divergence. Indeed, if $\vec{F} = (x, y, z)$, then $\nabla.\vec{F} = 3$ whereas if $\vec{F} = (-y, x, 0)$, then $\nabla.\vec{F} = 0$. Again, these examples are too naive. The divergence is more subtle as we shall see later on. Just as $\nabla \times \nabla f = \vec{0}$, one can easily prove (HW) that $\nabla.(\nabla \times \vec{F}) = 0$. This "easy" observation lead Maxwell to add a corection term (called the displacement current) to Ampere's law. $\Delta f$ (or $\nabla^2 f$) defined by $\Delta f = \nabla.(\nabla f) = f_{xx} + f_{yy} + f_{zz}$ is called the Laplacian of $f$ and plays a major role in electrostatics ( among other things).

Akin to Stokes' theorem, we have a generalisation of FTC to three-space.
Theorem: Let $V$ be a solid in $\mathbb{R}^3$ bounded by a closed regular surface $S$ parametrised

with the outward unit normal. If $\vec{F}$ is a $C^1$ vector field on $V$, then $\int\int\int_V \nabla.\vec{F}dxdydz = \int\int_S \vec{F}.d\vec{A}$. So the flux integral can be written as a triple integral.
Proof: Again, it suffices to prove it for $\vec{F} = (P, 0, 0)$. However, the proof is quite tricky in general. We shall prove it only for a cuboid. (The same proof works for Type-III regions.) $\int_e^f \int_c^d \int_a^b P_x dxdydz = \int\int (P(b, y, z) - P(a, y, z))dydz$. Now $\vec{F}.d\vec{A} = 0$ for the boundary sides that are not parallel to the $y - z$ plane. Thus, the flux is $\int\int P\hat{i}.d\vec{A}$. The boundaries are oriented in opposite directions and hence we are done.

Example: Compute the outward flux of $\vec{F} = (x, y, z)$ across the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$: One way is to take $(a\sin(\theta)\cos(\phi), b\sin(\theta)\sin(\phi), c\cos(\theta))$, compute $d\vec{A} = \vec{r}_\theta \times \vec{r}_\phi d\theta d\phi$ which equals $(bc\sin^2(\theta)\cos(\phi), ac\sin^2(\theta)\sin(\phi), ab\sin\theta\cos\theta)d\theta d\phi$, compute $\vec{F}.d\vec{A} = abc\sin(\theta)$ and integrate.
The smart way is to use the divergence theorem: $\nabla.\vec{F} = 3$. Hence the answer is $4\pi abc$.
Another example: Let $\vec{F} = e^{-(x^2+y^2+z^2)^6}(x, y, z)$. Compute $\int\int\int_V \nabla.\vec{F}dV$ where $V$ is the unit ball. It is of course quite painful to do directly. However, using the divergence theorem, it is the flux of a radial vector field over the unit sphere. Thus it is $4\pi e^{-1}$.

Interpretation of Divergence: The divergence is the flux density, i.e., near a point $p$, $\nabla.\vec{F}(p)$ is approximately the ratio of the outward flux through a small closed surface divided by its volume. Because of this subtle interpretation, counterintuitive things like the following can happen:
Example: The divergence of $\vec{F} = (-y(x^2 - y^2), x(x^2 - y^2), 0)$ is non-zero.
Example: The divergence of $\vec{F} = \frac{1}{(x^2+y^2+z^2)^{3/2}}(x, y, z)$ is zero.