1 Recap

- Line integrals. Reparametrisation invariance.

2 Multiple integrals

To evaluate volumes, fluxes, areas, etc, we need to develop multivariable integration and a fundamental theorem of calculus. Roughly speaking, if $f(\vec{r})$ is a scalar field on a "rectangular" domain, $[a_1, b_1] \times [a_2, b_2] \times \ldots$, the integral $\int\int\int f(x_1, \ldots)dV$ ought to be defined as the sum over all small rectangles. Hopefully, this definition will coincide with the more naive definition of integrating one variable at a time (multiple integrals vs iterated integrals). To this end, we shall define/work with two variables, but everything generalises word-to-word to more variables.
Def: A partition P of $[a, b] \times [c, d]$ is a subset $P_1 \times P_2$ such that $P_1 = \{x_0 = a, \ldots, x_n = b\}$ and $P_2 = \{y_0 = c, \ldots, y_m = d\}$ are partitions of $[a, b]$ and $[c, d]$ respectively. A partition gives rise to a bunch of open subrectangles. A partition P' is said to be *finer* than P if $P \subset P'$. Given any two partitions, their union is finer than both and is called a *common refinement.*
Def: A function defined on a rectangle Q is said to be a step function if there exists a partition on whose corresponding open subrectangles, the function is a constant.
It is easy to show that if s_1, s_2 are step functions on Q, then $c_1s_1+c_2s_2$ is a step function on Q. Thus, step functions form a real vector space. The volume of a cuboid is $Area \times height$. Hence we define: Let f be a step function that takes c_{ij} on $(x_{i-1}, x_i) \times (y_{j-1}, y_j) \subset Q$. Then the double integral of f over Q is defined by $\sum_{i,j} c_{ij}\Delta x_i \Delta y_j$.

- As in 1-D, one can prove that this value is *independent* of the partition chosen. We denote the sum as $\int\int_Q f(x, y)dA$ and call it the double integral of f over Q. It is easy to prove that this double integral equals the *iterated* integrals $\int_a^b \int_c^d x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy$.
- Linearity: $\int\int_Q (c_1s_1 + c_2s_2)dA = c_1 \int\int_Q s_1dA + c_2 \int\int_Q s_2dA$.
- Additivity: If Q is divided into two rectangles that intersect only in their sides, then $\int\int_Q sdA = \int\int_{Q_1} sdA + \int\int_{Q_2} sdA$.
- Comparison: If $s \leq t$ on Q, then $\int\int sdA \leq \int\int tdA$.

Let $f : Q \to \mathbb{R}$ be a bounded function, i.e., $|f| \leq M$ on Q. Clearly, there exist step functions s, t such that $s \leq f \leq t$ on Q.
Def: If there exists a unique number I such that $\int\int_Q s \leq I \leq \int\int_Q t$ for every pair of step functions s, t such that $s \leq f \leq t$, then I is called the double integral of f over Q and is denoted as $\int\int_Q fdA$. If such an I exists, f is said to be Riemann-integrable over Q.
Let S be the supremum of all numbers $\int\int_Q s$ where s is a step function such that $s \leq f$, and likewise, T be the infimum of $\int_Q t$ where $f \leq t$. Then $\int\int_Q s \leq S \leq T \leq \int\int_Q t$ for all $s \leq f \leq t$.

Thus f is R.I over Q if and only if $S = T$. T is called the upper integral and S the lower integral. As in $1D$, and as for step functions in $2D$, the additivity, linearity, and comparison theorems continue to hold.

Fubini theorem for rectangles: Let $f : Q \to \mathbb{R}$ be bounded and integrable (this assumption is crucial!) Except for finitely many values, assume that $g(y) = \int_a^b f(x,y)dx$ and $h(x) = \int_c^d f(x,y)dy$ exist, and are integrable over $[c,d]$ and $[a,b]$ respectively. Then $\int\int_Q f dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$.
Proof: Let $s \leq f \leq t$. Integrate w.r.t x on both sides (valid by assumptions and $1D$-properties). Then integrate w.r.t y. By definition, we are done. $\square$
This theorem allows us to calculate double integrals and geometrically interpret the double integral as the volume under a graph. Indeed, $g(y)$ is the area of a cross-section and its integral is the volume.