1 Recap

- Multiple integrals. Fubini for rectangles.

2 Multiple integrals

Continuous functions are double integrable and Fubini holds:
Sketch of proof: By the extreme value theorem, f is bounded. Thus the lower (S) and upper (T) integrals exist. It is enough to come up with a sequence of "special" step functions $s_n \leq f \leq t_n$ so that $\int\int_Q s_n dA \leq S \leq T \int\int_Q t_n dA$ converge to the same quantity. We choose a sequence of partitions P_n such that the variation of $f \to 0$ as $n \to \infty$ (by continuity). The special step functions are simply the infimum and supremum functions $m_n(x,y), M_n(x,y)$. Since f_n does not vary much, m_n, M_n are close to each other and hence so are their integrals. $\square$
To prove that the integrals equal their iterated versions, it is enough to prove that $g(y) = \int_a^b f(x,y)dx$ and $h(x) = \int_c^d f(x,y)dy$ are continuous. This follows from some estimates.

Discontinuous functions: Integrals of discontinuous beasts are problematic even in 1-D. However, if we have only finitely many discontinuities in 1D, we can integrate.

Akin to that, if the set of discontinuities in $2D$ have "zero area" (whatever that means), the function is still integrable (proof is skipped). For instance, it turns out that (proof skipped) a finite collection of line segments or more generally, a finite collection of C^1 regular paths have zero area. In particular, $(x, f(x))$ or $(g(y), y)$ where f, g are C^1 have zero area.

3 Non-rectangular domains

If Ω is a bounded region, i.e., it is contained in some rectangle Q, then extend f to $\tilde{f}$ on Q by setting it to 0 outside Ω.
Def: A bounded function f is said to be integrable over Ω if $\int\int_\Omega f dA := \int\int_Q \tilde{f} dA$ exists. One can prove that this definition makes sense, i.e., a different choice of Q does not change anything. The real problem is whether one can prove that continuous functions f are integrable and whether Fubini holds. For this, it is crucial that the boundary of Ω (the place where $\tilde{f}$ can be discontinuous) is of zero area. Surely this is the case for Type-I domains: $a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)$ where ϕ_1, ϕ_2 are C^1 on $[a,b]$, and Type-II domains: $c \leq y \leq d, \psi_1(y) \leq x \leq \psi_2(y)$ where ψ_1, ψ_2 are C^1 on $[c,d]$.

Let S be a Type-I region, i.e., $a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)$. Assume that $f : S \to \mathbb{R}$ is bounded and continuous on the interior. Then $\int\int_S f$ exists and equals $\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dydx$. (Likewise, for Type-II. If it is both of Type-I and Type-II, it is sometimes called Type-III and in that case, the iterated integrals are equal.)
Proof: Recall that the boundary of a Type-I domain has zero area. Thus the extension $\tilde{f}$ to $[a,b] \times [c,d]$ is integrable. Moreover, for each x, $\int_c^d \tilde{f}(x,y)dy$ is integrable because there

are at most two discontinuities. Thus by Fubini, $\int\int_S f := \int\int_Q \tilde{f} = \int_a^b \int_c^d \tilde{f}(x,y)dydx = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dydx$.

Examples:

- Firstly, if S is a Type-I domain, then $\int\int_S dA = \int_a^b(\phi_2(x) - \phi_1(x))dx$ is indeed the Area. Likewise, $\int\int\int_S dV$ is the volume of a Type-I domain in 3-D. The same results hold for Type-II domains as well.

- Example 1: Calculate the volume of an ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. An ellipsoid is a Type-I domain in 3-D: $-c\sqrt{1 - x^2/a^2 - y^2/b^2} \leq z \leq c\sqrt{1 - x^2/a^2 - y^2/b^2}$. Thus the volume is $\int\int\int dV = 2c\int\int \sqrt{1 - x^2/a^2 - y^2/b^2}dA$ over $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$. Now the latter is a Type-I domain in $2D$: $-b\sqrt{1 - x^2/a^2} \leq y \leq b\sqrt{1 - x^2/a^2}$. Thus the volume is $2bc\int_{-a}^{a}\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} \sqrt{1 - x^2/a^2 - y^2/b^2}dydx$. Now $A(x) = \int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} \sqrt{1 - x^2/a^2 - y^2/b^2}dy$ can be evaluated easily by substitution (or by geometry) and found to be $b\frac{\pi}{2}(1 - \frac{x^2}{a^2})$. Now the final volume is $bc\int_{-a}^{a} \pi(1 - \frac{x^2}{a^2})dx = \frac{4}{3}\pi abc$.

- Example 2: Suppose f is continuous on a bounded region S and $\int\int_S f dA = \int_0^3 \int_{4y/3}^{\sqrt{25-y^2}} f(x,y)dxdy$. then sketch S and interchange the order of integration: $\frac{4y}{3} \leq x \leq \sqrt{25 - y^2}$. Hence it is the region between a circle and a line passing through the centre. $x = \frac{4y}{3}$ and $x = \sqrt{25 - y^2}$ intersect at $(4,3)$. Thus we have two regions $0 \leq y \leq \frac{3x}{4}$ in $0 \leq x \leq 4$ and $0 \leq x \leq \sqrt{25 - y^2}$ in $4 \leq x \leq 5$. So the integral is $\int_0^4 \int_0^{3x/4} f dydx + \int_4^5 \int_0^{\sqrt{25-y^2}} f dydx$.