1 Recap

- Non-rectangular domains. Fubini. Triple integrals.

2 Non-rectangular domains

Examples:

- Example 3: Calculate $\int\int_{[-1,1]\times[0,2]} \sqrt{|y-x^2|}dA$. We integrate over y first: $\int_{-1}^{1}(\int_0^{x^2} \sqrt{x^2-y}dy + \int_{x^2}^{2} \sqrt{y-x^2}dy)$. Since x is a constant, we can easily integrate to get $\frac{2}{3}x^3 + \frac{2}{3}\left(2-x^2\right)^{3/2}$. The second term can be evaluated by trigonometric substitution. So we get $\frac{4}{3}+\frac{\pi}{2}$. Note that first integrating over x would have made life worse.

3 What should an FTC look like?

We want to formulate a fundamental theorem of calculus. In 1-D, recall that it is $\int_a^b f'(x)dx = f(b)-f(a)$. In other words, "the integral of a derivative over a region boils down to its boundary". So in $2D$, there are a few questions: What regions must we consider? If $f(x,y)$ is a scalar field, what "derivative" must we integrate? and since the boundary is a curve, what must the integral boil down to (presumably a line integral on the boundary)? For the first question, we must choose a domain whose boundary is a piecewise C^1 regular curve (to make sense of line integrals). Furthermore, the region must not have "holes" because then the boundary can be more than one curve (such regions are called simply connected). It turns out that every simple closed curve divides $\mathbb{R}^2$ into two regions the interior region is simply connected (Jordan curve theorem).

Theorem: Let P, Q be C^1 scalar fields on a simply connected closed set S whose boundary is a piecewise C^1 regular curve. Then $\int_C (Pdx + Qdy)$ taken in the anti-clockwise direction equals $\int\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$.
The proof is quite painful and is frankly, beyond the scope of this course. However, let us look at a special case of a rectangle: The boundary is piecewise C^1 and is parametrised as (t,c) $a \leq t \leq b$, (b,t) $c \leq t \leq d$, $(-t,d)$ $-b \leq t \leq -a$, and $(a,-t)$ $-d \leq t \leq -c$. Thus the line integral is $\int_a^b P(t,c)dt - \int_{-b}^{-a} P(-t,d)dt + \int_c^d Q(b,t)dt - \int_{-d}^{-c} Q(a,-t)dt$. By the usual FTC, this equals the other side of Green's theorem. It is not hard to do the same thing for Type-III domains (HW). In the general case, one approximates the boundary by a many-sided polygon and breaks the interior of this polygon up into a bunch of rectangles and triangles (all Type-III). Then one applies the above proof to each and adds.

Examples:

- Find the area of the circle $x^2+y^2=1$. The area is $\int\int dxdy$. Choose $Q=x, P=-y$ and use Green: $2\times Area = \int_C (xdy - ydx)$. Parametrise C as $x=\cos(t), y=\sin(t)$. Thus $xdy - ydx = dt$. Thus $Area = \pi$. (A device called the planimeter works on this principle!)

- Consider $\int \frac{ydx-xdy}{x^2+y^2}$ over the circle of radius r centred at $(0,0)$. Parametrise it as $(r\cos(t), r\sin(t))$, $0 \leq t \leq 2\pi$. Then the integral is -2π. However, naively applying the Green theorem, we get 0 !! What is going wrong? The point is that the domain of P, Q is actually the disc minus the origin, i.e., it has a hole. So Green is not applicable! (This way of deducing the shape of regions by what kind of calculus one can do on them is a big thing. It is called "Differential Topology".)

Change of variables (a digression from FTCs): In one-variable calculus, recall that if $u : [a,b] \to [c,d]$ is a $1-1$ onto C^1 map such that $\frac{du}{dx} \neq 0$ for all x (except possibly at a, b), and $f : [c,d] \to \mathbb{R}$ is continuous function, then $\int_{u^{-1}(a)}^{u^{-1}(b)} f(u(x))u'(x)dx = \int_c^d f(u)du$. Examples: Take $\int_0^2 u^2du$ with $u = 2x$ to get $\int_0^1 8x^2dx$. But with $u = 2-2x$, $\int_0^2 u^2du = -\int_1^0 8(1-x)^2dx = 2\int_0^1 4(1-x)^2dx$. If we want to use only $[0,2]$ and $[0,1]$ instead of the upper and lower limits, then in the second example, $\int_{[0,2]} u^2du = \int_{[0,1]} 4(1-x)^2|\frac{du}{dx}|dx$. We want a generalisation for multiple integrals. (Indeed, if we have cylindrical or spherical symmetry, it makes sense to choose other coordinates systems like polar coordinates.)