1 Recap

- Green's theorem and examples.

2 What should an FTC look like?

Change of variables (a digression from FTCs): Suppose we change coordinates $(u, v) \to (x(u, v), y(u, v))$. Then, the infinitesimal area element dA in (x, y) is a small parallelogram with vertices $(x(u, v), y(u, v))$, $(x(u+du, v), y(u+du, v)) = (x(u, v)+\frac{\partial x}{\partial u}du, y(u, v)+\frac{\partial y}{\partial u}du)$, etc. The tiny side-vectors of the parallelogram are $du(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u})$ and $dv(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v})$. Thus the area is $dA = dudv|\vec{r}_u \times \vec{r}_v|$ which is $dudv|J(u, v)|$ where $J(u, v)$ is called the Jacobian and is the *determinant* of the derivative matrix. Note that one takes the modulus of J and thus no sign appears. The region's shape of course changes in new coordinates. Another important point is that none of the infinitesimal parallelogram's should be "crushed" to lines or points because we want change-of-variables to "preserve" information. Thus, morally, we expect $J \neq 0$ to be a natural assumption.

Let $\Omega, \tilde{\Omega} \subset \mathbb{R}^2$ be bounded open sets. Let $f : \tilde{\Omega} \to \mathbb{R}$ be a continuous bounded function. Let $(x(u, v), y(u, v)) : \Omega \to \tilde{\Omega}$ be a C^1 $1-1$ onto map such that $J(x(u, v), y(u, v)) \neq 0$ everywhere. Then $\int\int_{\Omega} f(x(u, v), y(u, v))|J(u, v)|dudv = \int\int_{\Omega} f(x, y)dxdy$. A similar statement holds in higher dimensions too.
The proof is surprisingly complicated. We shall prove a special case later on. Calculate $\int\int_{\mathbb{R}^2} e^{-(x^2+y^2)}dxdy$ (I am cheating by using improper integrals). Choose polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$. $J = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = r$. Thus $\int_0^{\infty}\int_0^{2\pi} e^{-r^2}rd\theta dr = 2\pi\int_0^{\infty} e^{-r^2}rdr = \pi$. Let $I = \int_{-\infty}^{\infty} e^{-x^2}dx$ and hence $I^2 = \pi$. This is the easiest way to evaluate the Gaussian integral. (One way to justify these things is to say $I = \lim_{a\to\infty}\int_{-a}^{a} e^{-x^2}dx = \lim_{a\to\infty}\int_{-a}^{a} e^{-y^2}dy$ and hence $I^2 = \lim_{a\to\infty}\int_{-a}^{a}\int_{-a}^{a} e^{-(x^2+y^2)}dxdy$ by limit laws. Now using an easy inequality we can prove that the difference between this integral and the integral over a disc over radius a goes to 0 as $a \to \infty$. Hence we use change of variables and conclude that $I^2 = \lim\int_0^{a}\int_0^{2\pi} e^{-r^2}rdrd\theta$ and so on.)

Let S be the triangle bounded by $x+y = 2$ and the axes. Evaluate $\int\int_S e^{(y-x)/(y+x)}dxdy$ (cheating again by improper integrals). Let $u = y - x, v = y + x$. Then the triangle is bounded between $u + v = 0 = u - v$, $v = 2$. The modulus of the Jacobian is $|J| = \frac{1}{2}$. So we integrate $\frac{1}{2}\int_0^2\int_{-v}^{v} e^{u/v}dudv$ which is $\frac{1}{2}\int_0^2 v(e - e^{-1})dv = e - e^{-1}$.

Proof in a special case: Assuming the change of variables is C^2, we shall prove $\int\int_R dxdy = \int\int_{\tilde{R}} |J|dudv$ i.e. when $f = 1$, and $\tilde{\Omega} = \tilde{R}$ is a rectangle using Green's theorem. Assume WLOG that $J > 0$. $\int\int_R dxdy = \int_C xdy$. Likewise, note that the RHS is $\int\int_{\tilde{R}} \frac{\partial}{\partial u}(x\frac{\partial y}{\partial v}) - \frac{\partial}{\partial v}(x\frac{\partial y}{\partial u})$. Thus by Green it is $\int\int_{\tilde{C}}(x\frac{\partial y}{\partial v})du + x\frac{\partial y}{\partial v}dv)$. Suppose we parametrise $\tilde{C}$ as $(u(t), v(t))$. Then $(x(u(t), v(t)), y(u(t), v(t)))$ is a parametrisation for the rectangle. Its velocity is $(x_u u' + x_v v', y_u u' + y_v v')$. Using the change of parametrisation formula, we are done. One can use this special case to prove the general case.

Calculate the volume $V_n(a)$ of an n-dimensional ball $x_1^2 + x_2^2 \leq +x_n^2 \leq a^2$.
Firstly, we prove that $V_n(a) = a^n V_n(1)$: Let $x = au$ where u is a part of a unit ball. Then $J = a^n$ and the change of variables formula does the trick. As for $V_n(1)$, it equals the iterated integral $\displaystyle\int_{x_{n-1}^2+x_n^2\leq 1} \int\int \ldots \int_{x_1^2+x_2^2+\ldots x_{n-2}^2\leq 1-x_{n-1}^2-x_n^2} dx_1 \ldots dx_{n-2}dx_{n-1}dx_n$. Now the inner integrand is $V_n(\sqrt{1-x_{n-1}^2-x_n^2}) = (1-x_{n-1}^2-x_n^2)^{(n-2)/2}V_{n-2}(1)$. Thus $V_n(1) = V_{n-2}(1)\int\int_D(1-x^2-y^2)^{(n-2)/2}dxdy = 2\pi V_{n-2}(1)\int_0^1(1-r^2)^{(n-2)/2}rdr = V_{n-2}(1)\frac{2\pi}{n}$. We can calculate using this formula.