# 1 Recap

- Existence of determinants.
- $\det(A) = \det(A^T)$, row operations.
- Calculting using Gauss-Jordan.

# 2 The product formula

If $A, B$ are two $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$. This formula is *extremely* important.
Proof: Denote the $i^{th}$ column of of $B$ by $B_i$. Recall that the $i^{th}$ column of $(AB)$, i.e., $(AB)_i$ is $AB_i$. Thus $\det((AB)_1, (AB)_2, (AB)_3, \ldots) = \det(AB_1, AB_2, \ldots)$. Fix $A$ and define $F(B_1, \ldots, B_n) = \det(AB_1, AB_2, \ldots)$. Note that $F$ is

1. multilinear: $F(\ldots, tB_i + sv, \ldots) = \det(\ldots, A(tB_i + sv), \ldots) = \det(\ldots, tAB_i + sAv, \ldots)$ which is $t\det(\ldots, AB_i, \ldots) + s\det(\ldots, Av, \ldots)$ and hence $F(\ldots, tB_i + sv, \ldots) = tF(\ldots, B_i, \ldots) + sF(\ldots, v, \ldots)$.
2. alternating: $F(\ldots, B_i = v, \ldots, B_j = v, \ldots) = \det(\ldots, Av, \ldots, Av, \ldots) = 0$.

Hence by uniqueness, $F(B_1, \ldots, B_n) = \det(B_1, \ldots, B_n)F(e_1, \ldots, e_n)$. Thus $\det(AB) = \det(B)\det(A)$.

# 3 Invertibility and determinants

If an $n \times n$ matrix $A$ is invertible then $AA^{-1} = I$. Thus $\det(A)\det(A^{-1}) = 1$ and hence $\det(A) \neq 0$.
Recall that if a set of $n$ vectors $v_1, \ldots, v_n$ from $\mathbb{F}^n$ is linearly dependent, then $\det(v_1, \ldots, v_n) = 0$. Thus, if $\det(A) \neq 0$, its columns are linearly independent and hence its column rank is full. Thus $A$ is invertible.
Therefore, $A$ is invertible if and only if $\det(A) \neq 0$. Equivalently, the set $v_1, \ldots, v_n$ is independent if and only if $\det(v_1, \ldots, v_n) \neq 0$.

# 4 Block-diagonal matrices

Let $A$ be an $n \times n$ matrix and $D$ be an $m \times m$ matrix. Then the matrix $M = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ is an $(n+m) \times (n+m)$ "block diagonal" matrix. Note that $M = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}$. So $\det(M)$ is a product of two other determinants. The function $F(A_1, \ldots, A_n) = \det \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$ satisfies all the axioms of multilinear alternating functions and hence by uniqueness, $F(A_1, \ldots) = \det(A)F(e_1, \ldots) = \det(A)$. Likewise for $D$. Thus $\det(M) = \det(A)\det(D)$. $\square$

# 5 Change of basis

Given a linear map $T : V \to V$ where as always, $V$ is a f.d vector space, how can we define its determinant?
One naive thing to do is to consider an ordered basis $e_1, \ldots, e_n$ for both, the domain and the target. Then $T$ is represented by a matrix $[T]$. We can attempt to define $\det(T)$ as $\det([T])$. However, what happens when we change the ordered basis? Let $e'_1, e'_2, \ldots$ be a new ordered basis. Recall that $e'_i = \sum_j P_{ji} e_j$ for some $P_{ji} \in \mathbb{F}$. Also recall that in an ordered basis the first column of the matrix associated to $T$ is simply the component vector of $T(e'_1)$ and likewise for the other columns. Now $T(e'_i) = \sum_j P_{ji} T(e_j)$ which is $\displaystyle\sum_{j,k} P_{ji}[T]_{kj} e_k$.
We wish to express $T(e'_i)$ in terms of the $e'$s (as opposed to $e$s). In your HW you will prove that $P$ is invertible and that $[T]' = [P]^{-1}[T][P]$. Two $n \times n$ matrices $A, B$ are said to be *similar* if there is an invertible matrix $P$ such that $B = P^{-1}AP$. We just proved that if we change ordered bases using an invertible matrix $P$ (whose columns represent the new basis) then $[T]$ and $[T]'$ are similar. One can prove the converse too (HW). Now one can see that $\det([T]') = \det([P]^{-1}) \det([T]) \det([P]) = \det([T])$. Hence the determinant of a linear map can be defined by choosing *any* ordered basis.