# 1 Recap

- Proved that $\det(AB) = \det(A)\det(B)$ and computed determinants of block diagonal matrices.
- $A$ is invertible iff $\det(A) \neq 0$.
- If we choose different bases for expressing $T : V \to V$, then $[T]' = [P]^{-1}[T][P]$ where $[P]$ is an invertible matrix called the change of basis matrix. Conversely, if two matrices are similar, then they define the same linear map from $V$ to itself.

# 2 Powers of a matrix

The Fibonacci sequences is $F_0 = 0$ $F_1 = 1$ $F_2 = 1 + 0 = 1$ $F_n = F_{n-1} + F_{n-2}$. It was discovered by the Indians in the context of Sanskrit poetry (!) much earlier. Fibonacci himself discovered them whilst modelling rabbit populations. They turn up unexpectedly in CS, maths (Hilbert's tenth problem), and supposedly in biology. Is there a formula for $F_n$? Consider $F_n = F_{n-1} + F_{n-2}$ and $F_{n-1} = F_{n-1}$ (!). So $F_n, F_{n-1}$ are *linear* in $F_{n-1}, F_{n-2}$. So $\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}$. Let $v_n = \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$, and $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Then $v_n = Mv_{n-1}$. Thus $v_n = M^{n-2}v_2$. How does one write a formula for $M^{n-2}$?

Suppose the chance that it rains tomorrow if it rains today is 0.7 and the chance that it rains tomorrow if it does not rain today is 0.5. What is the chance that it rains after 10 days given that it rained today? If the chance that it rains after $n$ days is $p_n$ and $q_n = 1 - p_n$ is the chance that it does not rain, then $p_n = 0.7p_{n-1} + 0.5q_{n-1}$ and $q_n = 0.3p_{n-1} + 0.5q_{n-1}$. Moreover, $p_0 = 1 = 1 - q_0$ ( because it rained today). We want $p_{10}$. As in the case of Fibonacci numbers if $v_n = \begin{bmatrix} p_n \\ q_n \end{bmatrix}$, then $v_n = Mv_{n-1}$ where $M = \begin{bmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{bmatrix}$. So what is $M^n$? ( This simple model is an example of a Markov chain.)

If $A$ is an $n \times n$ matrix, when is it "easy" to calculate $A^n$? It is so if $A$ is diagonal. In that case, $A^n = diag(a_{11}^n, a_{22}^n, \ldots)$. Given an arbitrary $n \times n$ $A$ is there a "natural" way to relate it to a diagonal matrix? One natural way to change matrices is through similarity, i.e., $B = P^{-1}AP$. Note that $B^n = P^{-1}A^nP$, i.e., $A^n = PB^nP^{-1}$. So if $B$ is diagonal then we are in good shape. In the language of linear maps, given a linear map $T : V \to V$, is there a basis $e_1, \ldots, e_n$ such that the matrix corresponding to $T$, i.e., $[T]$ is diagonal? That is, $T(e_1) = \lambda_1 e_1$, $T(e_2) = \lambda_2 e_2$, etc?

# 3 Eigenvalues and eigenvectors

Def: Let $T : V \to V$ be a linear map and $V$ be a vector space. A *non-zero* vector $v$ is said to be an *eigenvector* with *eigenvalue* $\lambda \in \mathbb{F}$ if $Tv = \lambda v$.
Rookie mistake: An eigenvector by definition is required to NOT be the zero vector!
Theorem: If $V$ is a f.d. vector space, and $T : V \to V$ is linear then $[T]$ is diagonal in

an ordered basis $e_1, \ldots$ if and only if the basis vectors $e_1, \ldots, e_n$ are eigenvectors with eigenvalues $[T]_{11}, [T]_{22}, \ldots$.
Proof:

- If $[T]$ is diagonal: Clearly $[T][e_i] = [T]_{ii}[e_i]$. Hence $Te_i = T_{ii}e_i$, i.e., $e_i$ are eigenvectors.

- If $e_i$ are eigenvectors with eigenvalues $\lambda_1, \lambda_2, \ldots$: $Te_i = \lambda_i e_i$. By definition of $[T]$, it is diagonal with diagonal entries $\lambda_i$.

A linear map $T : V \to V$ is said to be diagonalisable if it has a basis of eigenvectors. Likewise, a square matrix $A$ is said to be diagonalisable if there is an invertible matrix $P$ such that $P^{-1}AP$ is diagonal.

Examples:

- Consider $T : V \to V$ given by $T(v) = cv$ for all $v \in V$. *Every* non-zero vector is an eigenvector with eigenvalue $c$. Eigenspace: Suppose $T(v) = \lambda v$ for a non-zero $v$ then $T(cv) = cT(v) = \lambda(cv)$ for *every* $c \in \mathbb{F}$. Moreover, if $T(w) = \lambda w$, then $T(v + w) = T(v) + T(w) = \lambda(v + w)$. In other words, the set of all eigenvectors corresponding to the *same* eigenvalue along with the zero vector forms a *subspace* known as the *eigenspace* of $\lambda$.

- Consider the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(v) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Then $T(e_1) = 0$. Hence $T$ *can* have 0 as an eigenvalue without $T$ being zero itself! In this example $T(v) = \lambda v$ precisely when $y = \lambda x$, $0 = \lambda y$. Thus $\lambda = 0$ is the *only* eigenvalue and the eigenspace is spanned by one vector, i.e., it is one-dimensional. Thus $T$ is *not* diagonalisable.

- Consider rotation in $\mathbb{R}^2$ by 90 degrees. Clearly, there is *no* non-zero vector $v$ such that $Tv = \lambda v$ where $\lambda \in \mathbb{R}$. The matrix in the standard basis is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Note that $\begin{bmatrix} \sqrt{-1} \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\sqrt{-1}$. So the field is important.

- Suppose $V$ is the space of $f : \mathbb{R} \to \mathbb{R}$ having derivative of all orders ( 'smooth' functions). $D : V \to V$ given by $D(f) = f'$ is a linear map. $Df = \lambda f$ precisely when $f'(x) = \lambda f(x)$. As we shall prove later, $f(x) = Ae^{\lambda x}$ is the solution.

# 4 A criterion for eigenvalues

Suppose $V$ is f.d. If $v$ is an eigenvector of $T : V \to V$ with eigenvalue $\lambda$ then $(\lambda I - T)v = 0$ where $I : V \to V$ is $I(x) = x$, and $v \neq 0$. Therefore $N(\lambda I - T) \neq 0$. Thus $\lambda I - T$ is NOT invertible. Hence $\det(\lambda I - T) = 0$. That is, if $e_1, \ldots, e_n$ is an ordered basis, and $[T]$ is the corresponding matrix, $\det(\lambda[I] - [T]) = 0$. Conversely, if $\det(\lambda I - T) = 0$, then there exists a non-zero $v$ such that $Tv = \lambda v$. Therefore, eigenvalues are precisely solutions to $p_T(\lambda) = \det(\lambda I - T) = 0$ lying in $\mathbb{F}$. One can prove (HW) using induction that $p_T(\lambda)$

is a polynomial of degree $n$ with highest power being $\lambda^n$ and $p_T(0) = \det(0 - T) = (-1)^n \det(T)$. This polynomial is called the *characteristic polynomial* of $T$. Eigenvalues

- Firstly, not every real polynomial has real roots!
- However, it is an important result that every complex polynomial has at least one complex root. ( The fundamental theorem of algebra.) Moreover, if it has degree $n$ then counting roots with multiplicity, there are exactly $n$ roots.
- Thus every complex matrix has $n$ complex eigenvalues when counted with multiplicity.
- Assume from now that every vector space is over $\mathbb{C}$ and so is every matrix.